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Pion condensation in effective field theories

Master's thesis

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Abstract

In this thesis, we investigate the phase diagram of dense quark matter at finite temperature. A finite isospin chemical potential is included to allow for charged pion condensation. This is done in the framework of two different effective theories for QCD.

We study the O(N) linear sigma model to leading order in the 1/N expansion, for both zero and nonzero vacuum masses of the pions. We find that pion condensation occurs at low temperatures for isospin chemical potentials larger than the pion mass, and that the transition to this phase is second order.

The phase diagram of the two-flavour Nambu–Jona-Lasinio model is investigated in the chiral limit, and here we also include a finite baryon number chemical potential. It is found that pion condensation occurs at low temperatures, and that chiral symmetry is restored at high temperatures. The model is also used to study neutral matter, such as one might find in the core of compact stars.

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Chapter 1

Introduction

Suppose that we take a lump of matter and squeeze it, very, very hard...

Matter consists, under ordinary circumstances, of atoms. An atom is, as we know, a positively charged nucleus surrounded by a cloud of negatively charged electrons. The nucleus is again made up from protons and neutrons (collectively called *nucleons*), each of which is a combination of three quarks. The largest part of an atom, however, is just a whole lot of empty space.

If we squeeze our lump of matter so hard that the nuclei of adjacent atoms bump into each other, and even harder, what happens then? At some point the average matter density will become larger than the density of the nuclei, and protons and neutrons will start to overlap. Suddenly it is no longer possible to distinguish one nucleon from another, and our lump has become what is called *quark matter*. In this state of matter it is, at not too high temperatures, expected that quarks and antiquarks will pair up to form positive and negative *pions*, a phenomenon known as *pion condensation*.

The dominant interaction between particles in the quark matter state is the strong interaction, described by quantum chromodynamics (QCD). Unfortunately, as will be made clear later in this chapter, QCD can be a notoriously difficult theory to work with. Therefore, one often has to resort to other models, models which in some way can be used as an approximation to QCD. In this thesis we will study pion condensation in the framework of two such models, the *linear sigma model* and the *NJL model*.

This thesis is organised as follows: In the remainder of this chapter we give a short introduction to symmetries, symmetry breaking and quantum chromodynamics. In chapter 2 we introduce the necessary mathematical tools, most notably the merging of quantum field theory with statistical mechanics known as *thermal field theory*. In chapter 3 we study pion condensation in the linear sigma model, whereas the chapters 4 and 5 are devoted to the NJL model. A lot of the work done in making this thesis has been numerical calculations on a computer. The appendices B and C therefore contain the source code for some of the computer programs used to perform these calculations.

1.1 Notation

Before we begin, we summarise the most important notations and conventions used in this thesis.

- We always work in natural units, so $\hbar = c = k_{\rm B} = 1$.
- Ordinary three-vectors are written in bold face, \mathbf{v} , or with Roman indices, v^i . Four-vectors are written with Greek indices $v^{\mu} = (v^0, \mathbf{v})$. Repeated indices are summed over, unless otherwise is explicitly specified.
- We will use a metric tensor with signature g = (+, -, -, -) in Minkowski space and g = (+, +, +, +) in Euclidean space.
- The Pauli matrices are denoted by the letter τ when acting in flavour space. They are defined

 \mathbf{as}

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.1)$$

and are often collected in a vector, $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$.

• The Dirac matrices are defined by the anticommutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \tag{1.2}$$

and the Hermiticity conditions

$$(\gamma^{\mu})^{\dagger} = \gamma^0 \gamma^{\mu} \gamma^0. \tag{1.3}$$

A fifth anticommuting γ -matrix, γ^5 , is defined by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.4}$$

and has the properties

$$\{\gamma^{\mu}, \gamma^{5}\} = 0, \qquad (\gamma^{5})^{2} = 1, \qquad (\gamma^{5})^{\dagger} = \gamma^{5}.$$
 (1.5)

Greek indices always take the values 0, 1, 2, 3 only, and never 5.

• We denote the trace of a matrix (the sum of its diagonal elements) with

$$\operatorname{tr} A \equiv \sum_{n} A_{nn},\tag{1.6}$$

while the trace of an operator (the sum of the operator's diagonal matrix elements) is denoted with the 't' capitalised:

$$\operatorname{Tr} \mathcal{O} \equiv \sum_{n} \langle n | \mathcal{O} | n \rangle.$$
(1.7)

The trace has the cyclic and additive properties

$$tr ABC = tr CAB \tag{1.8}$$

$$\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B. \tag{1.9}$$

• Real scalar fields are denoted by the small Greek letter phi (ϕ), while complex scalar fields are denoted by a capital phi (Φ). Dirac spinors are written as a small psi (ψ).

1.2 Symmetry

The study of symmetries is an integral part of modern physics. It was the discovery of E. Noether [1] that a continuous symmetry leads to a conserved quantity, and vice versa. By symmetry we mean *invariance of the equations describing a system with respect to some transformation*. For instance, if the Lagrangian, or Hamiltonian for that matter, of a system is invariant under a rotation of the coordinate system about some axis, we have rotational symmetry about that axis. By Noether's theorem, this implies conservation of angular momentum. By the same token, invariance of a system under a temporal or spatial translation implies conservation of energy or linear momentum, respectively.

We will now look at some of the most important symmetries in quantum field theory, and subsequently we will investigate what happens when symmetries are broken.

1.2.1 Noether's theorem in field theory

Suppose that the fields ψ_a depend on the set ϵ_r of continuous parameters. Then Noether's theorem states that the four-current

$$j_r^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi_a)} \frac{\delta\psi_a}{\delta\epsilon_r},\tag{1.10}$$

where $\delta \epsilon_r$ denotes a small variation of the parameter ϵ_r , is conserved [2]. This means that

$$\partial_{\mu}j_{r}^{\mu} = 0. \tag{1.11}$$

We can also associate a conserved charge with this current. By integrating this equation over three-dimensional space and applying Gauss' divergence theorem to the term containing the spatial current, we get

$$0 = \int \mathrm{d}^3 x \; \partial_\mu j_r^\mu = \int \mathrm{d}^3 x \; \left(\frac{\partial j_r^0}{\partial t} + \nabla \cdot \mathbf{j}_r\right) = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 x \; j_r^0 + \int_S \mathbf{j}_r \cdot \mathrm{d}\mathbf{S}. \tag{1.12}$$

We define the charge

$$Q_r \equiv \int \mathrm{d}^3 x \; j_r^0, \tag{1.13}$$

and assume that \mathbf{j}_r falls off sufficiently fast as \mathbf{x} goes to infinity that the surface integral is zero. With this definition, the charge density is $\rho_r(x) \equiv j_r^0(x)$. Eq. (1.12) then becomes

$$\frac{\mathrm{d}Q_r}{\mathrm{d}t} = 0,\tag{1.14}$$

showing that Q_r is constant in time.

1.2.2 Real scalar fields and O(N) symmetry

Scalar fields, both real and complex, describe spin-zero bosons. Assume that we have a set of N free (non-interacting) real scalar fields $\phi_1, \phi_2, \ldots, \phi_N$, with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \frac{1}{2} m^2 \phi_i \phi_i.$$
(1.15)

Let R be an orthogonal $N \times N$ matrix, that is, a matrix with the property $R^T R = 1$ or, equivalently, $R_{ij}R_{ik} = \delta_{jk}$. We apply the following transformation to the fields:

$$\phi_i \to R_{ij}\phi_j. \tag{1.16}$$

It is then easily shown that the Lagrangian is invariant under this transformation:

$$\mathcal{L} \to \frac{1}{2} R_{ij}(\partial_\mu \phi_j) R_{ik}(\partial^\mu \phi_k) - \frac{1}{2} m^2 R_{ij} \phi_j R_{ik} \phi_k = \frac{1}{2} (\partial_\mu \phi_i) (\partial^\mu \phi_i) - \frac{1}{2} m^2 \phi_i \phi_i.$$
(1.17)

Eq. (1.16) is called a global O(N) transformation. The O stands for *orthogonal* and N is the number of rows and colums in the transformation matrix. *Global* just means that the transformation does not depend on the position in spacetime—the components of R are constants. Since the theory is invariant under this transformation, we say that we have a global O(N) symmetry.

We note that under this transformation, $\phi_i \phi_i \to \phi_i \phi_i$, so the length of the vector ϕ_i is preserved. An O(N) transformation is therefore a *rotation* in \mathbb{R}^N . Strictly speaking, the *special* orthogonal group, SO(N), which is the group of orthogonal $N \times N$ matrices with determinant equal to one, is what is usually called the rotation group [3]. O(N) does not preserve the orientation of the coordinate axes.

1.2.3 Complex scalar fields and U(1) symmetry

Next, assume that we have a set of N complex scalar fields $\Phi_1, \Phi_2, \ldots, \Phi_N$, with Lagrangian

$$\mathcal{L} = (\partial_{\mu}\Phi_i)^*(\partial^{\mu}\Phi_i) - m^2\Phi_i^*\Phi_i.$$
(1.18)

It is easily shown that by defining $\Phi_i = (\phi_{2i-1} + i\phi_{2i})/\sqrt{2}$, Eq. (1.18) is the same as Eq. (1.15) with 2N real fields. The theory is therefore O(2N) symmetric.

We now apply the phase transformation

$$\Phi_i \to e^{i\alpha} \Phi_i, \qquad \Phi_i^* \to \Phi_i e^{-i\alpha}, \qquad \alpha = \text{const},$$
(1.19)

called a global U(1) transformation. This transformation is $unitary^1$, hence the U. The Lagrangian then transforms as

$$\mathcal{L} \to (\partial_{\mu}\Phi_{i})^{*}e^{-i\alpha}e^{i\alpha}(\partial^{\mu}\Phi_{i}) - m^{2}\Phi_{i}^{*}e^{-i\alpha}e^{i\alpha}\Phi_{i} = (\partial_{\mu}\Phi_{i})^{*}(\partial^{\mu}\Phi_{i}) - m^{2}\Phi_{i}^{*}\Phi_{i},$$
(1.20)

so this is a symmetry of the theory.

To find the Noether current associated with this symmetry, we assume that the above transformation is infinitesimal, so that

$$\Phi_i \to (1+i\alpha)\Phi_i, \qquad \Phi_i^* \to (1-i\alpha)\Phi_i^*,$$
(1.21)

that is, we have $\delta \Phi_i / \delta \alpha = i \Phi_i$ and $\delta \Phi_i^* / \delta \alpha = -i \Phi_i^*$. Differentiating the Lagrangian with respect to the derivatives of the fields, we obtain

$$\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \Phi_{i})} = (\partial^{\mu} \Phi_{i})^{*}, \qquad \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \Phi_{i})^{*}} = \partial^{\mu} \Phi_{i}.$$
(1.22)

The conserved current density is then, according to Eq. (1.10),

$$j_i^{\mu} = i \left(\Phi_i \partial^{\mu} \Phi_i^* - \Phi_i^* \partial^{\mu} \Phi_i \right), \qquad (1.23)$$

where no summation over *i* is implied. Here we have $j_i^{\mu} = (n_i, \mathbf{j}_i)$, where n_i is the density of particles of type *i* and \mathbf{j}_i is the current density. A conserved current implies a conserved charge—in this case the particle number $N_i = \int d^3x n_i$.

1.2.4 Symmetries of a Dirac field

U(1) symmetry

A Dirac field, denoted by ψ , describes spin- $\frac{1}{2}$ fermions. The free Dirac Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi. \tag{1.24}$$

As in the previous section, we apply the global phase transformation

$$\psi \to e^{i\alpha}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha}, \qquad \alpha = \text{const},$$
 (1.25)

and see that the Lagrangian is U(1) symmetric:

$$\mathcal{L} \to \bar{\psi} e^{-i\alpha} (i\gamma^{\mu}\partial_{\mu} - m) e^{i\alpha} \psi = \bar{\psi} (i\gamma^{\mu}\partial_{\mu} - m) \psi.$$
(1.26)

According to Eq. (1.10), the conserved current is $(\bar{\psi}i\gamma^{\mu})(i\psi) + 0 \cdot (-i\bar{\psi}) = -\bar{\psi}\gamma^{\mu}\psi$. The sign is immaterial—a conserved quantity multiplied by a constant is still conserved—so we define

$$j^{\mu} = (n, \mathbf{j}) = \bar{\psi} \gamma^{\mu} \psi. \tag{1.27}$$

Again the conserved charge is the particle number, $N = \int d^3x n$. We may also define

$$j_{\rm em}^{\mu} = (\rho, \mathbf{J}) = qj^{\mu} = q\bar{\psi}\gamma^{\mu}\psi, \qquad (1.28)$$

¹A unitary matrix is a matrix with the property that $U^{\dagger}U = \mathbf{1}$, or, $U_{ij}^{*}U_{ik} = \delta_{jk}$.

where $\pm q$ is the electric charge of the fermions, ρ is the charge density and **J** is the electromagnetic current density. Then j_{em}^{μ} is the electromagnetic four-current, and the relation $\partial_{\mu} j_{\text{em}}^{\mu} = 0$ is the well-known *continuity equation* from classical electrodynamics:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$
(1.29)

The conserved charge is, of course, the electric charge:

$$Q = \int \mathrm{d}^3 x \ \rho = \int \mathrm{d}^3 x \ qn = qN. \tag{1.30}$$

Chiral symmetry

We define the *left-handed* and *right-handed* fields as

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi, \text{ and } \psi_R = \frac{1}{2}(1 + \gamma^5)\psi,$$
 (1.31)

so that

$$\psi = \psi_L + \psi_R$$
 and $\bar{\psi} = \bar{\psi}_L + \bar{\psi}_R$. (1.32)

It turns out that this is a useful decomposition, because ψ_L describes (anti-)fermions with negative (positive) helicity, and ψ_R describes (anti-)fermions with positive (negative) helicity.

We now write the Lagrangian in terms of these fields:

$$\mathcal{L} = [\bar{\psi}_L + \bar{\psi}_R](i\gamma^{\mu}\partial_{\mu} - m)[\psi_L + \psi_R]$$

$$= \bar{\psi}_L(i\gamma^{\mu}\partial_{\mu} - m)\psi_L + \bar{\psi}_R(i\gamma^{\mu}\partial_{\mu} - m)\psi_R$$

$$+ \bar{\psi}_L(i\gamma^{\mu}\partial_{\mu} - m)\psi_R + \bar{\psi}_R(i\gamma^{\mu}\partial_{\mu} - m)\psi_L.$$
(1.33)

From Eq. (1.31) we see that

$$\psi_L^{\dagger} = \frac{1}{2} \psi^{\dagger} [1 - (\gamma^5)^{\dagger}] = \frac{1}{2} \psi^{\dagger} (1 - \gamma^5), \qquad (1.34)$$

since $(\gamma^5)^{\dagger} = \gamma^5$. Thus,

$$\bar{\psi}_L = \psi_L^{\dagger} \gamma^0 = \frac{1}{2} \psi^{\dagger} (1 - \gamma^5) \gamma^0 = \frac{1}{2} \bar{\psi} (1 + \gamma^5), \qquad (1.35)$$

because of the anticommutativity of γ^0 and γ^5 . In the same manner,

$$\bar{\psi}_R = \frac{1}{2}\bar{\psi}(1-\gamma^5).$$
 (1.36)

Now look at the cross terms in \mathcal{L} . Note that

$$\bar{\psi}_L i \gamma^\mu \partial_\mu \psi_R = \frac{1}{2} \bar{\psi} (1+\gamma^5) i \gamma^\mu \partial_\mu \frac{1}{2} (1+\gamma^5) \psi$$

$$= \frac{1}{4} \bar{\psi} (1+\gamma^5) (1-\gamma^5) i \gamma^\mu \partial_\mu \psi$$

$$= \frac{1}{4} \bar{\psi} [1-(\gamma^5)^2] i \gamma^\mu \partial_\mu \psi$$

$$= 0,$$
(1.37)

since $(\gamma^5)^2 = 1$. In the same way, $\bar{\psi}_R i \gamma^\mu \partial_\mu \psi_L = 0$. We then obtain

$$\mathcal{L} = \bar{\psi}_L (i\gamma^\mu \partial_\mu - m)\psi_L + \bar{\psi}_R (i\gamma^\mu \partial_\mu - m)\psi_R - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L).$$
(1.38)

From Eq. (1.38) we see that the mass term is the only term that couples ψ_L and ψ_R . If m = 0 they decouple and can be treated as independent fields. The Lagrangian is then $U(1)_L \times U(1)_R$ invariant, that is, we may apply different phase transformations to ψ_L and ψ_R . This is called a *chiral symmetry*.

Any chiral symmetry can also be decomposed into two separate symmetries in another way: A vector symmetry, $U(1)_V$, which treats the left- and right-handed fields equally, and an axial symmetry, $U(1)_A$, which treats them differently. The former is merely the symmetry we described in the previous section, while the latter is defined as invariance under the transformation

$$\psi \to e^{i\alpha\gamma^5}\psi, \qquad \alpha = \text{const.}$$
 (1.39)

Note that

$$\bar{\psi} \to \bar{\psi} e^{i\alpha\gamma^5},$$
 (1.40)

because γ^5 in the exponent anticommutes with γ^0 in $\bar{\psi}$, thus cancelling the minus sign from complex conjugation. The Lagrangian then transforms as

$$\mathcal{L} \rightarrow \bar{\psi} e^{i\alpha\gamma^5} (i\gamma^{\mu}\partial_{\mu} - m) e^{i\alpha\gamma^5} \psi = \bar{\psi} e^{i\alpha\gamma^5} e^{-i\alpha\gamma^5} i\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi} e^{i\alpha\gamma^5} e^{i\alpha\gamma^5} \psi = \bar{\psi} i\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi} e^{i2\alpha\gamma^5} \psi.$$

$$(1.41)$$

and, as expected, we see that \mathcal{L} remains invariant under the axial transformation only in the limit m = 0, called the *chiral limit*.

Flavour symmetry

We now take ψ to be a quark field with two flavours (*up* and *down*), and write it as a two-spinor in flavour space:

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{1.42}$$

By analogy with ordinary particle spin, **S**, we introduce the concept of *isospin*, **I**. The field ψ is said to carry isospin $I = \frac{1}{2}$, and the third component of the isospin, I_3 , has the eigenvalues $+\frac{1}{2}$ (the up quark) and $-\frac{1}{2}$ (the down quark).

We then apply the transformations

$$\psi_L \to e^{i\alpha_i^L \tau_i} \psi, \qquad \psi_R \to e^{i\alpha_i^R \tau_i} \psi$$
(1.43)

where α_i^L and α_i^R (i = 1, 2, 3) are arbitrary constants. In the chiral limit it is easily seen that the Lagrangian (1.38) is invariant under these transformations:

$$\mathcal{L} \to \bar{\psi}_L e^{-i\alpha_i^L \tau_i} i\gamma^\mu \partial_\mu e^{i\alpha_i^L \tau_i} \psi_L + \bar{\psi}_R e^{-i\alpha_i^R \tau_i} i\gamma^\mu \partial_\mu e^{i\alpha_i^R \tau_i} \psi_R = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R \quad (1.44)$$

The exponential factors are 2×2 unitary matrices with determinant 1, and therefore belong to the SU(2) group. The symmetry group of the above transformation is then $SU(2)_L \times SU(2)_R$, yet another example of a chiral symmetry.

Again, we can decompose the symmetry into vector and axial parts, related to the transformations

$$\psi \to e^{i\alpha_i \tau_i} \psi, \qquad \psi \to e^{i\theta_i \gamma^5 \tau_i} \psi,$$
(1.45)

respectively. If we give nonzero and equal mass to the two quark flavors, $m_u = m_d \neq 0$, the chiral symmetry is broken and we are left with $\mathrm{SU}(2)_V$. If $m_u \neq m_d$ this is broken down to separate U(1) symmetries for the two flavours, $\mathrm{U}(1)_{V,u} \times \mathrm{U}(1)_{V,d}$. The SU(2) flavour symmetry is commonly called *isospin* symmetry.

The analogy with spin is lost if we generalise to N_f flavours, where $N_f > 2$. In that case, the symmetry group of the flavour symmetries is $SU(N_f)$.

1.2.5 Gauge symmetry

So far, all the symmetries we have considered have been global symmetries. But what happens when we *gauge* a symmetry, that is, make it dependent of position in spacetime? As the simplest example of this, we again consider the free fermion Lagrangian (1.24), and apply the *local* transformation

$$\psi \to e^{i\alpha(x)}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha(x)}.$$
 (1.46)

It is easy to show that because the kinetic term in the Lagrangian involves differentiation with respect to x, the Lagrangian is *not* invariant under this transformation. To mend this, one has to introduce the *gauge covariant derivative*,

$$\partial_{\mu} \to D_{\mu} = \partial_{\mu} + iqA_{\mu}(x),$$
 (1.47)

where $A_{\mu}(x)$ is a bosonic vector field—a vector gauge boson—and q is again the electric charge of the fermions. To ensure invariance under the transformation (1.46), the field A_{μ} is then required to transform as

$$A_{\mu}(x) \to A_{\mu}(x) - \frac{1}{q} \partial_{\mu} \alpha(x).$$
 (1.48)

Finally, we add the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, giving the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - q\bar{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$
(1.49)

The Euler-Lagrange equation for the field A_{μ} is then

$$\partial_{\mu}F^{\mu\nu} = j^{\nu}_{\rm em},\tag{1.50}$$

where j_{em}^{ν} is given by Eq. (1.28). By closer inspection one finds that these are exactly the inhomogeneous Maxwell's equations for the electromagnetic vector potential A_{μ} . Therefore, the procedure described here is one way of deriving the quantum theory of the electromagnetic interaction, quantum electrodynamics, or QED for short.

Although more complicated, the above procedure has been used to derive other theories as well. We will see an example of this in Sec. 1.4.

1.3 Symmetry breaking

There are two ways a symmetry of a Lagrangian density may be broken. The obvious one, and also the least interesting one, is *explicit breaking*: adding (or removing) terms in the Lagrangian so that a certain symmetry no longer exists.

A far more interesting situation occurs when a symmetry of the Lagrangian is *not* a symmetry of the ground state. This is called *spontaneous* symmetry breaking and is the topic of this section.

1.3.1 Goldstone's theorem

Goldstone's theorem is a famous theorem that relates spontaneously broken continuous symmetries to the appearance of new particles in the spectrum—the so called Goldstone bosons. In its most general form it can be stated as follows [4, 5, 6]:

If a continuous symmetry is spontaneously broken, there must be an excitation mode in the spectrum of the theory whose energy vanishes in the limit of zero momentum.

In this formulation the theorem applies to nonrelativistic as well as relativistic field theory.

Note that the theorem does not say anything about the number of different Goldstone bosons, nor does it say much about their properties. Nonrelativistic field theory will not be discussed in this thesis, but in relativistic field theory we know that if the energy goes to zero with the momentum, the particles must be massless. Also, in Lorentz invariant theories, the number of Goldstone modes is equal to the number of broken-symmetry generators [7]. The simpler and more well-known, but less general, form of the theorem is thus:

For each generator of a spontaneously broken continuous symmetry there will be a massless mode in the spectrum.

When the broken symmetry is not exact, but approximate, that is, if the symmetry is (explicitly) broken softly as well as spontaneously, the modes will become massive but light.



Figure 1.1: "Mexican hat" potential for N = 2.

1.3.2 Linear sigma model

To see an example of spontaneous symmetry breaking, we again turn to the model from Sec. 1.2.2, but this time we add an O(N)-symmetric self-interaction term:

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \frac{1}{2} m^2 \phi_i \phi_i - \frac{\lambda}{4} (\phi_i \phi_i)^2.$$
(1.51)

This is called the *linear sigma model*. In the following we will assume that $m^2 < 0$, and therefore replace the mass with a more general parameter μ by taking $\mu^2 = -m^2$.

We now identify the ground state, or vacuum state, as the state that minimises the classical potential

$$\mathcal{V}(\phi_i) = -\frac{1}{2}\mu^2 \phi_i \phi_i + \frac{\lambda}{4} (\phi_i \phi_i)^2.$$
(1.52)

This potential is shown in Fig. 1.1 for N = 2. The circular trough represents the continuous set of vacuum states.

The ground state is then the constant field $\phi_i = \phi_i^0$, where

$$\phi_i^0 \phi_i^0 = v^2, \qquad v \equiv \frac{\mu}{\sqrt{\lambda}}.$$
(1.53)

We see that Eq. (1.53) only determines the magnitude of ϕ_i^0 . All directions are equivalent, so we may use O(N) symmetry to choose coordinates such that ϕ_i^0 points in the Nth direction:

$$\phi_i^0 = (0, 0, \dots, 0, v). \tag{1.54}$$

We now assume that ϕ_i is near the vacuum state and look at small fluctuations around this state:

$$\phi_i = (\pi_k(x), v + \sigma(x)), \quad k = 1, 2, \dots, N - 1.$$
(1.55)

For the N = 2 case, we see that $\sigma(x)$ represents excitations in the radial direction, that is, in the direction where the potential energy changes. The $\pi_k(x)$, on the other hand, represents excitations in the angular direction—along the circular trough in Fig. 1.1.

Writing the Lagrangian in terms of the π_k and σ fields, we obtain

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \pi_{k})^{2} + \frac{1}{2} (\partial_{\mu} \sigma)^{2} - \frac{1}{2} (2\mu^{2})\sigma^{2} - \sqrt{\lambda}\mu\sigma^{3} - \sqrt{\lambda}\mu(\pi_{k})^{2}\sigma - \frac{\lambda}{4}\sigma^{4} - \frac{\lambda}{2} (\pi_{k})^{2}\sigma^{2} - \frac{\lambda}{4} [(\pi_{k})^{2}]^{2}, \qquad (1.56)$$

where we have omitted constant terms. We see that this Lagrangian density describes one massive field, σ , and N-1 massless fields, π_k —the Goldstone bosons. Furthermore, the original O(N)symmetry is broken, leaving us with O(N-1) symmetry that rotates the π_k -fields amongst themselves. The O(N) group has $\frac{1}{2}N(N-1)$ generators, so the difference in the number of generators



Figure 1.2: Colour-electric field between a quark and an antiquark.

between the groups O(N) and O(N-1) is exactly N-1, the same as the number of Goldstone modes.

Note that for the classical potential to take the Mexican hat form, the mass squared must be negative. Obviously, m can therefore not represent a physical mass. In chapter 3 the linear sigma model will be treated in detail, and there we will also see how m is related to actual particle masses.

1.4 Quantum chromodynamics

Quantum chromodynamics, abbreviated QCD, is regarded as a correct description of the interaction between quarks—the *strong force* [8]. Unfortunately, it can be notoriously difficult to work with, which is why one often has to resort to other theories, theories that in some aspects have the same properties, to gain information about QCD itself. In this section we will give a brief introduction to the theory, its history and some effective theories.

1.4.1 Quarks and gluons

As far as we know, the quarks and their antiparticles are, together with leptons, the most elementary particles. Six *flavours* of quarks have been discovered: The up, down, strange, charm, top and bottom quarks. These are again classified in terms of electric charge and *colour* charge. In units of the electron charge e, the u, c and t quarks have $Q = +\frac{2}{3}$, while the d, s and b quarks have $Q = -\frac{1}{3}$. Colour is the name given to the charge of the strong interaction. A quark comes in one of three colours—red, green or blue—whereas an antiquark is either antired, antigreen or antiblue.

The strong force is mediated by particles called *gluons* (because they "glue" the quarks together). These are massless electrically neutral vector gauge bosons that, unlike photons, actually participate in the interaction. This is because the gluons themselves carry colour charge, and it means that the gluons not only interact with quarks, but also with themselves.

QCD exhibits a property called *confinement*, meaning that quarks are confined in colour singlets. More intuitively, we can say that only colour neutral states are observed, that is, either combinations of one red, one green and one blue quark (baryons), or combinations of a quark and an antiquark which have "opposite" colours—red and antired, for instance (mesons). Calculations using *lattice QCD*—a numerical approximation to QCD in which the continuum of spacetime is replaced by a discrete lattice [9]—have shown the potential between a quark and an antiquark to be compatible with the *Cornell* potential [10, 11],

$$\mathcal{V}(r) = ar - \frac{\kappa}{r} + C. \tag{1.57}$$

This potential is seen as a sum of a long-range term which encodes the confinement and a shortrange Coulomb potential arising from one-gluon exchange [12]. C is an arbitrary negative constant that can be adjusted to fit experiment. For large r, we see that the energy cost of pulling the quarks apart becomes proportional to the distance between them. This means that if the quarks are to be completely separated, an infinite amount of energy is needed. The colour-electric field between a quark and an antiquark can be modeled as a *flux tube* [13] with constant radius, shown schematically in Fig. 1.2. Because of this, the constant a is called the *string tension*.

Another peculiar property of QCD is what is known as *asymptotic freedom*. This means that the coupling between quarks and gluons is not constant, but decreases as the energy scale increases. This is a consequence of the fact that gluons carry colour. In the same way that the bare electric charge of an electron is partially screened by electrically charged virtual particles, the colour charge of a quark is partially screened by virtual quarks. However, the self-interaction of the gluons give cause to a similar, but opposite and stronger effect: The field is *augmented*—an effect sometimes referred to as *antiscreening*. It means that the closer you get to a quark—that is, the higher the energy of an incoming particle—the less gluons add to the antiscreening effect, and the weaker the coupling becomes. However, if the number of quark flavours was higher than 16, the screening effect of the quarks would prevail and the theory would no longer be asymptotically free.

1.4.2 History

During the 1950s, experimentalists discovered a large number of particles called *hadrons*—so many that it was hard to believe these were the fundamental constituents of Nature. The hadrons were classified according to properties such as electric charge, isospin and strangeness, but an explanation for these properties had yet to be found.

In the early 1960s, Gell-Mann and Ne'eman invented the *eightfold way* to sort the hadrons into groups with similar properties and masses [14]. To explain this structure, Gell-Mann and Zweig proposed that the hadrons consisted of smaller particles, which they called *quarks*, and that the quarks had to come in three flavours—the up, down and strange quarks.

There was, however, one particle which could not be explained by this model: The Δ^{++} , consisting of three up quarks with parallel spins. Quarks are fermions, and as such, this combination should be forbidden by the Pauli exclusion principle. This problem was resolved by the proposal of a new SU(3) degree of freedom, which was later named colour.

We have already seen that QED can be derived from the requirement that the theory is U(1) gauge invariant. So what happens if we impose a SU(3) gauge symmetry on the free quark Lagrangian? This is what we will do in the following section.

1.4.3 QCD as a gauge theory

Let ψ be a quark field with three colour degrees of freedom,

$$\psi = \begin{pmatrix} r \\ g \\ b \end{pmatrix}. \tag{1.58}$$

We now require that the Lagrangian (1.24) is invariant under a local SU(3) transformation,

$$\psi \to e^{i\alpha^a(x)T^a}\psi,\tag{1.59}$$

where $\alpha^1(x), \alpha^2(x), \ldots, \alpha^8(x)$ are real functions and the eight matrices T^1, T^2, \ldots, T^8 are the generators of the SU(3) group. The usual representation of these are $T^a = \lambda^a/2$, where λ^a are the Gell-Mann matrices

$$\lambda^{1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{2} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^{4} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^{5} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\lambda^{6} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^{7} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^{8} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (1.60)$$

The lambda matrices are hermitian, traceless and non-commuting, that is,

$$(\lambda^a)^\dagger = \lambda^a, \tag{1.61}$$

$$\operatorname{tr}\lambda^a = 0, \qquad (1.62)$$

$$\begin{bmatrix} \lambda^a, \lambda^b \end{bmatrix} = f^{abc} \lambda^c, \tag{1.63}$$

where f^{abc} are totally antisymmetric constants ($f^{123} = -f^{213}$, etc.).



Figure 1.3: Three- and four-gluon vertices.

Particle	Mass
u	1.5 to $3.0~{\rm MeV}$
d	3 to 7 MeV
s	$95\pm25~{\rm MeV}$
c	$1.25\pm0.09~{\rm GeV}$
b	$4.20\pm0.07~{\rm GeV}$
t	$174.2\pm3.3~{\rm GeV}$

Table 1.1: The current quark masses [16].

As in Sec. 1.2.5 we replace the ordinary derivative ∂_{μ} in the Lagrangian with the covariant derivative $D_{\mu} = \partial_{\mu} + igT^{a}G^{a}_{\mu}$, thus introducing eight different gluon fields, $G^{1}_{\mu}, G^{2}_{\mu}, \ldots, G^{8}_{\mu}$ [15]. In addition we have to include a kinetic term for the gluons. The complete QCD Lagrangian is then written as

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - g\gamma^{\mu}T^{a}G^{a}_{\mu} - m)\psi - \frac{1}{4}G^{a}_{\mu\nu}G^{a\ \mu\nu}, \qquad (1.64)$$

where

$$G^{a}_{\mu\nu} = \partial_{\mu}G^{a}_{\nu} - \partial_{\nu}G^{a}_{\mu} + gf^{abc}G^{b}_{\mu}G^{c}_{\nu}.$$
 (1.65)

It is obvious that the field strength tensor $G^a_{\mu\nu}$ is a more complicated object than its electromagnetic counterpart $F_{\mu\nu}$. By substituting Eq. (1.65) into Eq. (1.64), we see that we get terms not only quadratic in G^a_{μ} , but also cubic and quartic. These give rise to the gluon vertices shown in Fig. 1.3 and thus account for the self-interactions of the gluons.

When we apply the transformation (1.59), \mathcal{L}_{QCD} transforms as

$$\mathcal{L}_{\text{QCD}} \rightarrow \bar{\psi} e^{-i\alpha^{a}(x)T^{a}} (i\gamma^{\mu}\partial_{\mu} - g\gamma^{\mu}T^{a}G^{a}_{\mu} - m)e^{i\alpha^{a}(x)T^{a}}\psi$$
$$= \bar{\psi} \left[i\gamma^{\mu}\partial_{\mu} + ig\gamma^{\mu}T^{a}\left(G^{a}_{\mu} + \frac{1}{g}\partial_{\mu}\alpha^{a}(x)\right) - m\right]\psi.$$
(1.66)

We now see that if we simultaneously transform the gluon fields so that

$$G^a_\mu \to G^a_\mu - \frac{1}{g} \partial_\mu \alpha^a(x),$$
 (1.67)

the Lagrangian is left invariant under the transformation. Eq. (1.67) then defines the transformation properties of the gluon fields.

1.4.4 Symmetries

Table 1.1 is a listing of the current quark masses, from lightest to heaviest. Note that the three heaviest are a lot heavier than the other three. So much, in fact, that the c, b and t in most applications are quietly ignored, since it takes an enormous amount of energy to create even a single particle. The u and d (and sometimes the s), on the other hand, are so light that to simplify calculations they are often taken to be massless.

In Sec. 1.2.4 we found the symmetry groups of a free fermion field. These are preserved in QCD, and as we saw in Sec. 1.4.3, we also have the additional colour symmetry. For the QCD



Figure 1.4: Phase diagram of QCD [18, 19].

Lagrangian with N_f flavours of massless quarks we then find the following symmetry group:

$$SU(3)_{colour} \times SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A,$$
(1.68)

For finite but equal quark masses, $m_u = m_b = \cdots$, the chiral symmetry is explicitly broken down to

$$SU(3)_{colour} \times SU(N_f)_V \times U(1)_V.$$
 (1.69)

Finally, if the quark masses differ, the flavour symmetry is completely broken down to separate U(1) symmetries for each flavour. In addition to the above mentioned symmetry groups, the QCD Lagrangian is also invariant under the usual discrete symmetries, C, P and T.

It is a well-established fact that the chiral symmetry of massless QCD is spontaneously broken in the vacuum state. One should therefore in this case, according to Goldstone's theorem, observe a spectrum of massless bosons—the Goldstone bosons. In the real world the quarks are not massless, so we do not have exact chiral symmetry. The u and d, however, are so light that we have *approximate* chiral symmetry. This again means that the aforementioned Goldstone bosons acquire a little mass and are called *pseudo*-Goldstone bosons; we know them as the three pions, π^+ , π^- and π^0 , which have masses of about 140 MeV.

Finally we note that the axial U(1) symmetry of the Lagrangian, which is preserved in the classical theory, is spontaneously broken in the quantum theory. This is called a *quantum anomaly* and is closely related to *instantons* [17], which are solutions to the equations of motion of the classical field theory on a Euclidean spacetime. Instantons will not be discussed further here, but we will need to take the breaking of the U(1)_A symmetry into account when constructing the NJL Lagrangian in chapter 4.

1.4.5 Phase diagram

Fig. 1.4 shows schematically our current understanding of the QCD phase diagram, obtained using lattice QCD and effective theories. As we will see later, the chemical potential—the abscissa of the diagram—is directly related to particle densities. A high chemical potential effectively means high density, and vice versa.

For low temperatures and densities, we are in the normal, confined phase with spontaneously broken (approximate) chiral symmetry—the world we live in (Fig. 1.5a). At high temperatures, it is expected that matter becomes deconfined and chiral symmetry is restored. This phase is called the *quark-gluon plasma* (Fig. 1.5b). In the lower right part of the diagram, at low temperatures and high densities, we find quark matter. As stated in the introduction to this chapter, this is matter that is squeezed so hard together that the baryons start to overlap, and again the quarks are deconfined (Fig. 1.5c).



Figure 1.5: A schematic illustration of the three main phases of QCD: (a) Normal, confined matter. (b) Deconfined quark-gluon plasma. (c) Deconfined quark matter.

In the quark matter phase it is expected that a Bose–Einstein condensate of positive and negative pions will form under certain circumstances. It has been shown that at zero temperature this happens only if the *isospin chemical potential*—a measure of the imbalance between the number densities of up and down quarks—is larger than the pion mass [20, 21, 22, 23, 24, 25].

The quark matter phase is in Fig. 1.4 divided into three main subphases. In the *two-flavour* colour superconducting (2SC) and colour-flavor locked (CFL) phases the quarks form Cooper pairs, just like electrons in ordinary superconductivity. This leads to superconductivity of colour charge [26].

This thesis deals with the third phase, denoted NQ for "normal quark matter".

1.4.6 Effective models

Scalar field theories

Our first example of an effective theory for QCD is the linear sigma model, already introduced in Sec. 1.3.2. One then takes advantage of the fact that the symmetry breaking pattern of the model is group theoretically indistinguishable from the chiral symmetry breaking pattern of QCD. The linear sigma model is treated in detail in chapter 3.

A similar theory is the *nonlinear* sigma model, in which the interaction is replaced by a constraint on the fields ϕ_i . It can be derived from the linear sigma model by letting the coupling go to infinity [27]. The model possesses the same symmetries as the linear sigma model, and can therefore also be used as a model for QCD.

The sigma models have other applications than as low-energy effective theories for QCD. The Higgs field in the Standard Model is described by a linear sigma model, and in cosmology the model can be used to describe the *inflaton*—the scalar field that controls the inflation of the universe at early times. In condensed matter physics the sigma models are used to model spin-spin interactions.

The MIT bag model

The MIT bag model was proposed in the 70s as a model for hadrons [28, 29, 30]. The hadrons are modeled as free or weakly interacting quarks, confined to a finite region of space—the "bag"—by a vacuum pressure $P_{\text{vac.}} = B$. B is called the *bag constant*. Confinement is thus put in by hand as a boundary condition, and is not a dynamical result of the theory. Stability of the hadrons results from balancing the pressure of the vacuum outside the bag with the pressure from the quarks inside. The theory says nothing about the origin of this vacuum pressure, and just treats B as a free parameter.

The NJL model

The Nambu–Jona-Lasinio (NJL) model, which is the topic of chapters 4 and 5, is in some respects complementary to the MIT bag model. Where the latter is a phenomenological model of confinement, the NJL model incorporates spontaneously broken chiral symmetry, but not confinement.

Unlike the scalar field theories described above, the NJL model is a fermionic theory in which quarks are the fundamental degrees of freedom.

Chapter 2

Preliminaries

In this chapter we will develop the methods and techniques that we will use in this thesis. It is organised as follows: We start by giving a short introduction to some important mathematical concepts and formulae, which we then use to develop the functional integral formalism of quantum field theory. Finally, we show how this formalism can be used to describe field theory at finite temperature and density.

2.1 Mathematics

We start off this section by introducing the notion of *Grassmann numbers*. We will later see that Grassmann algebra is essential for describing fermionic systems using the functional integral formalism. Then we describe the technique of *functional derivatives*, which is used to extract information, such as correlation functions and other expectation values, from functional integrals. Finally, we present some generalisations of the Gaussian integral which will be used frequently.

2.1.1 Grassmann algebra

Grassmann numbers are anticommuting numbers [31]. A Grassmann algebra of a single variable η is defined by the anticommutator

$$\{\eta, \eta\} = 0. \tag{2.1}$$

Therefore, $\eta^2 = 0$, and the most general function of η is

$$f(\eta) = a + b\eta, \tag{2.2}$$

as seen by making a Taylor expansion of any (differentiable) function. Grassmann integration is thus exceedingly simple, being defined by only two rules:

$$\int \mathrm{d}\eta = 0, \qquad (2.3)$$

$$\int \mathrm{d}\eta \ \eta = 1. \tag{2.4}$$

The first of these follows from the requirement that any integral should be invariant under a shift of the integration variable, $\eta \to \eta + \xi$, while the second is a convenient choice of normalisation.

More generally, we consider a Grassmann algebra of a set of N variables η_i and the paired set η_i^* (i = 1, 2, ..., N), defined by the anticommutators

$$\{\eta_i, \eta_j\} = \{\eta_i, \eta_j^*\} = \{\eta_i^*, \eta_j^*\} = 0.$$
(2.5)

The most general function of these variables is

$$f(\eta_{1}, \eta_{1}^{*}, \dots, \eta_{N}, \eta_{N}^{*}) = a + \sum_{i} a_{i}\eta_{i} + \sum_{i} b_{i}\eta_{i}^{*} + \sum_{i \neq j} a_{ij}\eta_{i}\eta_{j} + \sum_{i \neq j} b_{ij}\eta_{i}^{*}\eta_{i}^{*} + \sum_{i,j} c_{ij}\eta_{i}\eta_{j}^{*} + \dots + d\eta_{1}\eta_{1}^{*} \cdots \eta_{N}\eta_{N}^{*}, \qquad (2.6)$$

and integration of such a function over all variables gives

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_1^* \cdots \mathrm{d}\eta_N \mathrm{d}\eta_N^* f(\eta_1, \eta_1^*, \dots, \eta_N, \eta_N^*) = d.$$
(2.7)

2.1.2 Functionals and functional derivatives

A functional is a function that takes one or more functions as arguments, instead of variables. We denote a functional using square brackets, so that the functional F of the functions u(x) and v(x) is written

$$F[u,v]. (2.8)$$

A typical example of a functional would be

$$F[u,v] = \int \mathrm{d}x \ u(x)v(x). \tag{2.9}$$

In the same way that we take the derivative of a functional with respect to some variable, we may take the derivative of a functional with respect to some function [17]. The most basic rule of functional differentiation is

$$\frac{\delta}{\delta u(x)}u(y) = \delta(x-y), \qquad (2.10)$$

or, equivalently,

$$\frac{\delta}{\delta u(x)} \int dy \ u(y)v(y) = v(x). \tag{2.11}$$

This is the natural generalisation of the discrete expressions

$$\frac{\partial}{\partial u_i} u_j = \delta_{ij} \qquad \frac{\partial}{\partial u_i} \sum_j u_j v_j = v_i \tag{2.12}$$

to continuous functions. Likewise, all the other rules of functional differentiation are completely analogous to the rules of ordinary differentiation. The chain rule, for instance, says that

$$\frac{\delta}{\delta u(x)} F[G[u]] = \frac{\delta F[G[u]]}{\delta G[u]} \frac{\delta G[u]}{\delta u(x)}.$$
(2.13)

Thus, the equation

$$\frac{\delta}{\delta u(x)} e^{\int \mathrm{d}y \ u(y)v(y)} = e^{\int \mathrm{d}y \ u(y)v(y)} \cdot v(x) \tag{2.14}$$

is an example of how one can use the chain rule to take functional derivatives of composite functions.

2.1.3 Gaussian integrals

The ordinary Gaussian integral in one variable is given by the formula

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \, e^{-\frac{1}{2}ax^2 + bx} = \frac{1}{\sqrt{a}} e^{b^2/2a}, \qquad a > 0.$$
(2.15)

As we will see, functional integrals that correspond to free (non-interacting) theories in quantum mechanics and quantum field theory have the form of Gaussian integrals. We now present four frequently-used formulae which are proven more or less generally in Ref. [32]. We will not need them all in this thesis, but they are included here for the sake of completeness.

Let $\{x_1, x_2, \ldots, x_N\}$ be real variables, $\{J_1, J_2, \ldots, J_N\}$ real numbers and S an $N \times N$ real symmetric positive definite matrix. Then,

$$\int \frac{\mathrm{d}x_1}{\sqrt{2\pi}} \cdots \frac{\mathrm{d}x_N}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i S_{ij} x_j + J_i x_i} = \frac{1}{\sqrt{\det S}} e^{\frac{1}{2}J_i S_{ij}^{-1} J_j} = e^{-\frac{1}{2} \operatorname{tr} \ln S} e^{\frac{1}{2}J_i S_{ij}^{-1} J_j},$$
(2.16)

where the integrals run from $-\infty$ to ∞ . In the last equality we used the identity

$$\ln \det A = \operatorname{tr} \ln A, \qquad (2.17)$$

which holds for any matrix A.

Next, let z_i be a set of complex variables and J_i a set of complex numbers, where i = 1, ..., N, and let H be an $N \times N$ matrix with a positive Hermitian part. Then,

$$\int \left(\prod_{i=1}^{N} \frac{\mathrm{d}z_{i}^{*} \mathrm{d}z_{i}}{2\pi i}\right) e^{-z_{i}^{*}H_{ij}z_{j} + J_{i}^{*}z_{i} + J_{i}z_{i}^{*}} = \frac{1}{\det H} e^{J_{i}^{*}H_{ij}^{-1}J_{j}} = e^{-\operatorname{tr}\ln H} e^{J_{i}^{*}H_{ij}^{-1}J_{j}}.$$
 (2.18)

Now we wish to establish the analogous identities for Grassmann variables. Let η_i be Grassmann variables and let ξ_i be Grassmann numbers, where i = 1, ..., N. For an $N \times N$ symmetric (but no longer necessarily positive definite) matrix S,

$$\int \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_N \ e^{-\frac{1}{2}\eta_i S_{ij}\eta_j + \xi_i \eta_i} = \sqrt{\det S} \ e^{\frac{1}{2}\xi_i H_{ij}^{-1}\xi_j} = e^{\frac{1}{2}\operatorname{tr}\,\ln S} \ e^{\frac{1}{2}\xi_i H_{ij}^{-1}\xi_j}.$$
(2.19)

For any $N \times N$ matrix A we have

$$\int \left(\prod_{i=1}^{N} \mathrm{d}\eta_{i}^{*} \mathrm{d}\eta_{i}\right) e^{-\eta_{i}^{*}A_{ij}\eta_{j} + \xi_{i}^{*}\eta_{i} + \xi_{i}\eta_{i}^{*}} = (\det A) \ e^{\xi_{i}^{*}A_{ij}^{-1}\xi_{j}} = e^{\operatorname{tr}\,\ln A} \ e^{\xi_{i}^{*}A_{ij}^{-1}\xi_{j}}.$$
 (2.20)

Note that in these expressions, the determinant appears in the numerator instead of the denominator.

2.2 Functional integrals

One of the most interesting quantities in quantum physics is the *transition amplitude* from one quantum state to another,

$$\langle \phi_f | e^{-iHt} | \phi_i \rangle, \tag{2.21}$$

where $|\phi_i\rangle$ and $|\phi_f\rangle$ are the initial and final states, respectively, t is the length of the time interval during which the transition takes place, and H is the Hamilton operator. We will now develop one method of calculating this amplitude, namely the Feynman path integral.

We introduce the formalism by applying it to a simple system—a single quantum mechanical particle—and afterwards we will see how to use functional integrals in quantum field theory. Finally, we will show how one can extract other useful information from path integrals.

2.2.1 Quantum mechanics of a single particle

Let $H(\hat{p}, \hat{x})$ be the Hamilton operator of a single-particle system, given in terms of the momentum operator \hat{p} and the position operator \hat{x} . Furthermore, let $|p\rangle$ and $|x\rangle$ be the respective eigenstates of these operators, so that

$$\hat{p}|p\rangle = p|p\rangle, \qquad (2.22)$$

$$\hat{x}|x\rangle = x|x\rangle, \qquad (2.23)$$

where p is the momentum and x is the position of the particle.

Consider the expression

$$w_{I,F} \equiv \langle x_F | e^{-i(t_F - t_I)H(\hat{p}, \hat{x})} | x_I \rangle.$$
(2.24)

This is the probability amplitude to find the particle in the state $|x_F\rangle$ at time t_F if it was known to be in the state $|x_I\rangle$ at time t_I . Now we divide the interval (t_I, t_F) into M subintervals of length ε , so that

$$\varepsilon \equiv \frac{t_F - t_I}{M}.\tag{2.25}$$

This yields

$$w_{I,F} = \langle x_F | \underbrace{e^{-i\varepsilon H(\hat{p},\hat{x})} \cdots e^{-i\varepsilon H(\hat{p},\hat{x})}}_{M \text{ times}} | x_I \rangle.$$
(2.26)

Between each exponential factor we insert a complete set of states,

$$1 = \int \mathrm{d}x \, |x\rangle \langle x|, \qquad (2.27)$$

yielding

$$w_{I,F} = \int \left(\prod_{i=1}^{M-1} \mathrm{d}x_i\right) \langle x_F | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_{M-1} \rangle \langle x_{M-1} | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_{M-2} \rangle \cdots \langle x_1 | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_I \rangle, \quad (2.28)$$

where we use the notation $x_I = x_0$, $x_F = x_M$. We see that for each interval we integrate over all possible positions of the particle, so what we actually do here is integrate over all the paths the particle can possibly take from x_I to x_F during the interval $t_F - t_I$. This is therefore called a *path* integral.

For each matrix element we now insert a complete set of momentum eigenstates,

$$\langle x_i | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_{i-1} \rangle = \int \mathrm{d}p_i \ \langle x_i | p_i \rangle \langle p_i | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_{i-1} \rangle, \tag{2.29}$$

where the overlap between a momentum and a position eigenstate is given by

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx}.$$
(2.30)

It can be shown [32] that

$$e^{-i\varepsilon H(\hat{p},\hat{x})} =: e^{-i\varepsilon H(\hat{p},\hat{x})} :+ \mathcal{O}(\varepsilon^2),$$
(2.31)

where : ... : denotes normal ordering, which means that all the \hat{p} 's are moved to the left and all the \hat{x} 's are moved to the right in each term. This means that when we take $\varepsilon \to 0$, we can consider $e^{-i\varepsilon H}$ to be normal ordered, allowing \hat{p} to act on $\langle p_i |$ in Eq. (2.29), while \hat{x} acts on $|x_{i-1}\rangle$. This gives

$$\langle x_i | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_{i-1} \rangle = \int \mathrm{d}p_i \, \langle x_i | p_i \rangle \langle p_i | x_{i-1} \rangle e^{-i\varepsilon H(p_i, x_{i-1})} + \mathcal{O}(\varepsilon^2)$$

$$= \frac{1}{2\pi} \int \mathrm{d}p_i \, e^{ip_i(x_i - x_{i-1}) - i\varepsilon H(p_i, x_{i-1})} + \mathcal{O}(\varepsilon^2).$$

$$(2.32)$$

Note that the operators in the expression have all been replaced by c-numbers. Now assume that we are dealing with a non-relativistic particle, which has the Hamilton operator

$$H(\hat{p}, \hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}), \qquad (2.33)$$

where $V(\hat{x})$ is the potential energy. Inserting this into Eq. (2.32), we obtain an ordinary Gaussian integral in p_i , easily evaluated using Eq. (2.15):

$$\langle x_{i} | e^{-i\varepsilon H(\hat{p},\hat{x})} | x_{i-1} \rangle = \frac{1}{2\pi} \int dp_{i} \ e^{ip_{i}(x_{i}-x_{i-1})-i\varepsilon \frac{p_{i}^{2}}{2m}-i\varepsilon V(x_{i-1})} + \mathcal{O}(\varepsilon^{2})$$

$$= \sqrt{\frac{m}{2\pi i\varepsilon}} e^{i\varepsilon \left[\frac{m}{2\varepsilon^{2}}(x_{i}-x_{i-1})^{2}-V(x_{i-1})\right]} + \mathcal{O}(\varepsilon^{2}).$$
(2.34)

Taking the limit $M \to \infty$, which also means that $\varepsilon \to 0$, we obtain

$$w_{I,F} = \lim_{M \to \infty} \int \left(\prod_{i=1}^{M-1} \mathrm{d}x_i \sqrt{\frac{m}{2\pi i \varepsilon}} \right) e^{i\varepsilon \sum_{i=1}^{M-1} \left[\frac{m}{2\varepsilon^2} (x_i - x_{i-1})^2 - V(x_{i-1}) \right]} \\ = \int_{x_I, t_I}^{x_F, t_F} \mathcal{D}[x(t)] e^{i \int_{t_I}^{t_F} \mathrm{d}t \left[\frac{m}{2} \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 - V(x) \right]},$$
(2.35)

where we have used the notation

$$\frac{x_i - x_{i-1}}{\varepsilon} \quad \to \quad \frac{\mathrm{d}x}{\mathrm{d}t},\tag{2.36}$$

$$\varepsilon \sum_{i=1}^{M-1} \rightarrow \int_{t_I}^{t_F} \mathrm{d}t,$$
 (2.37)

$$\int \left(\prod_{i=1}^{M-1} \mathrm{d}x_i \sqrt{\frac{m}{2\pi i\varepsilon}}\right) \quad \to \quad \int_{x_I, t_I}^{x_F, t_F} \mathcal{D}[x(t)]. \tag{2.38}$$

It is important to bear in mind that this is just that – notation, and nothing more. The path x(t) is generally not continuous nor differentiable. Whenever the notation is unclear or ambiguous, one should return to the discrete case to carry out calculations.

To summarise, we have obtained the expression

$$w_{I,F} = \langle x_F | e^{-i(t_F - t_I)H(\hat{p}, \hat{x})} | x_I \rangle = \int_{x_I, t_I}^{x_F, t_F} \mathcal{D}[x(t)] \ e^{iS[x(t)]}, \tag{2.39}$$

where

$$S[x(t)] = \int_{t_I}^{t_F} dt \ L[x(t)], \qquad (2.40)$$

$$L[x(t)] = \frac{m}{2} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x(t)). \qquad (2.41)$$

S is called the *action* and L is the system's Lagrangian. S and L are both functionals of a path x(t). $w_{I,F}$ is therefore called a *functional integral*, and denotes, as previously stated, the sum over all possible paths for the particle [33].

2.2.2 Quantum field theory

The preceding results can be readily extended to field theory, where we wish, for example, to find the transition amplitude

$$\langle \phi_F | e^{-i(t_F - t_I)H} | \phi_I \rangle \tag{2.42}$$

for a system in the state $|\phi_I\rangle$ at time t_I to be in the state $|\phi_F\rangle$ at time t_F . For detailed discussions on how to do this, see for example Refs. [31] and [32]; we will only present the results here. What we do, generally, is to replace the path x(t) with fields, $\phi(\mathbf{x}, t)$, and integrate over all "paths" the system can take from $|\phi_I\rangle$ to $|\phi_F\rangle$, that is, all possible field configurations of the system during the transition. In field theory one normally uses the Lagrangian *density* $\mathcal{L}[\phi(\mathbf{x}, t)]$ instead of L, so the action is written

$$S[\phi(\mathbf{x},t)] = \int_{t_I}^{t_F} \mathrm{d}t \ L = \int_{t_I}^{t_F} \mathrm{d}t \int \mathrm{d}^3x \ \mathcal{L}[\phi(\mathbf{x},t)], \qquad (2.43)$$

It is common to take the limit $t_I \to -\infty$, $t_F \to \infty$, making the action an integral over all of spacetime, $S = \int d^4x \mathcal{L}$, where x now denotes a four-vector in Minkowski space.

For a scalar field $\phi(x)$ we write the functional integral as

$$\langle \phi_F | e^{-i(t_F - t_I)H} | \phi_I \rangle = \int_{\phi_I, t_I}^{\phi_F, t_F} \mathcal{D}[\phi] \; e^{iS[\phi]}, \tag{2.44}$$

where S is given by Eq. (2.43) and the theory is described by the Lagrangian

$$\mathcal{L}[\phi] = \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \mathcal{V}(\phi).$$
(2.45)

The measure, as far as it can be called well-defined, is defined in a manner similar to the singleparticle case, Eq. (2.38),

$$\int_{\phi_I, t_I}^{\phi_F, t_F} \mathcal{D}[\phi] \equiv \lim_{M \to \infty} C \int \prod_{i=1}^{M-1} \mathrm{d}\phi_i, \qquad (2.46)$$

where C is an irrelevant constant.

For fermions we get the integral

$$\langle \psi_F | e^{-i(t_F - t_I)H} | \psi_I \rangle = \int_{\psi_I, t_I}^{\psi_F, t_F} \mathcal{D}[\bar{\psi}, \psi] \; e^{iS[\bar{\psi}, \psi]}. \tag{2.47}$$

Since $\bar{\psi}$ and ψ are anticommuting spinors, the latter has to be interpreted as an integral over Grassmann variables.

2.2.3 Correlation functions

We will now show how correlation functions, such as the propagators

$$\langle 0|T\phi(x)\phi(y)|0\rangle$$
 and $\langle 0|T\bar{\psi}(x)\psi(y)|0\rangle$, (2.48)

can be conveniently computed using functional integrals. We will demonstrate the technique using a scalar field theory, but the procedure for fermions is completely analogous.

It can be shown [17] that the time-ordered vacuum expectation value of an operator $\mathcal{O}(\phi)$ is given by the expression

$$\langle 0|T\mathcal{O}(\phi)|0\rangle = \frac{\int \mathcal{D}[\phi] \mathcal{O}(\phi) e^{iS[\phi]}}{\int \mathcal{D}[\phi] e^{iS[\phi]}},\tag{2.49}$$

where T is the time ordering operator. One can therefore write

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \frac{\int \mathcal{D}[\phi] \ \phi(x)\phi(y) \ e^{iS[\phi]}}{\int \mathcal{D}[\phi] \ e^{iS[\phi]}}.$$
(2.50)

Note that T does not appear on the right hand side of these expressions, since time ordering is incorporated by definition in the functional integral formalism.

We now define the generating functional W[J] as

$$W[J] \equiv \int \mathcal{D}[\phi] \ e^{iS[\phi] + \int d^4x \ J(x)\phi(x)}, \qquad (2.51)$$

where the function J(x) is an external *source*, coupled to the field ϕ . Using Eq. (2.14), the correlation functions can then be found by taking functional derivatives of W[J] with respect to J:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \frac{1}{W_0} \left. \frac{\delta}{\delta J(x)} \left. \frac{\delta}{\delta J(y)} W[J] \right|_{J=0},$$
(2.52)

where $W_0 = W[J = 0]$. Each functional derivative brings down a factor of ϕ in the integral, and by setting J = 0 we recover Eq. (2.50).

The action for the free theory is

$$S = \int d^4x \, \left[\frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2 \right].$$
 (2.53)

We integrate by parts to get

$$S = \frac{1}{2} \int d^4 x \ \phi(-\partial^2 - m^2)\phi, \tag{2.54}$$

where we have assumed that ϕ goes fast enough to zero as $x \to \infty$ so that the surface term vanishes. The notation is $\partial^2 \equiv \partial_\mu \partial^\mu$. The trick is now to interpret the operator $(-\partial^2 - m^2)$ as a *matrix* with continuous indices. We did the exact same thing in Sec. 2.1.2, when we generalised from a discrete vector u_i to a function u(x). One can then use Eq. (2.16) to write the generating functional in another way:

$$W[J] = \int \mathcal{D}[\phi] e^{\frac{1}{2} \int d^4 x \ \phi(x) [i(-\partial^2 - m^2)] \phi(x) + \int d^4 x \ J(x) \phi(x)}$$

= $C e^{-\frac{i}{2} \int d^4 x \ J(x) (-\partial^2 - m^2)^{-1} J(x)},$ (2.55)



Figure 2.1: (a) Tree-level propagator. (b) First-order loop correction. (c) Second-order loop correction.

where $C \equiv e^{-\frac{1}{2} \text{Tr } \ln[i(-\partial^2 - m^2)]}$. We will see how to calculate such operator traces later; for now the constant C can be ignored as it will cancel with the denominator in Eq. (2.52). Indeed, $W_0 = C$.

How do we then interpret an expression like $(-\partial^2 - m^2)^{-1}$? To complete the generalisation of a matrix, remember that for a matrix M and its inverse M^{-1} , the following holds: $M_{ij}^{-1}M_{jk} = \delta_{ik}$. Defining $D \equiv (-\partial^2 - m^2)^{-1}$, we therefore require that

$$(-\partial^2 - m^2)D(x - y) = \delta^{(4)}(x - y).$$
(2.56)

To solve this equation, we Fourier transform to momentum space,

$$(-\partial^2 - m^2) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, e^{ip(x-y)} D(p) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, e^{ip(x-y)},\tag{2.57}$$

which shows that

$$D(p) = \frac{1}{p^2 - m^2}.$$
(2.58)

Transforming back, we obtain

$$D(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, \frac{e^{ip(x-y)}}{p^2 - m^2 + i\varepsilon},\tag{2.59}$$

where the term $i\varepsilon$ is infinitesimal and added by hand to keep the integral from hitting a pole [34]. Eq. (2.55) can then be written as

$$W[J] = Ce^{-\frac{i}{2}\int d^4x \int d^4y J(x)D(x-y)J(y)},$$
(2.60)

and by using Eq. (2.52), we find

$$\langle 0|T\phi(x)\phi(y)|0\rangle = -iD(x-y). \tag{2.61}$$

As any textbook on quantum field theory will confirm, D(x - y) is indeed the propagator for a free scalar field. The operator $(-\partial^2 - m^2)$ is often called the *inverse propagator*, denoted D^{-1} . In momentum space we then have $D^{-1} = p^2 - m^2$.

It should be noted that the propagators we find using the free theory are *bare*, or *tree-level*, propagators, illustrated by the Feynman diagram in Fig. 2.1a. "Tree-level" means that loop corrections, examples of which are shown in Figs. 2.1b and 2.1c, that arise from interaction terms, are not included.

Calculating *n*-point correlation functions for n > 2 using this method is now easy; one just has to take more functional derivatives in Eq. (2.52). For the free theory above, for instance, the three-point correlator turns out to be zero, while the four-point function is

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = \frac{1}{W_0} \left. \frac{\delta}{\delta J_1} \left. \frac{\delta}{\delta J_2} \left. \frac{\delta}{\delta J_3} \left. \frac{\delta}{\delta J_4} W[J] \right|_{J=0} \right|_{J=0} = -i(D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}), \quad (2.62)$$

where $\phi_1 \equiv \phi(x_1)$, $D_{12} \equiv D(x_1 - x_2)$, et cetera. This corresponds to the sum of the three diagrams shown in Fig. 2.2.



Figure 2.2: Feynman diagram representation of the terms in the tree-level four-point correlation function.

2.3 Finite temperature and density

All the results we have obtained so far have been in vacuum. When we have calculated the propagator for a particle, for instance, we have assumed that there are no other particles present, or that their energy is so low that their contribution to the propagator is negligible. In the language of statistical physics this means that the temperature and particle densities are zero. But what happens when this is not the case? How does one include the effects of finite temperature and density? In this section we will merge quantum field theory with statistical physics, to obtain what is called *thermal field theory*.

2.3.1 The partition function

The central quantity in statistical physics is the *partition function* of a system. From this quantity one can extract all the usual thermodynamical quantities such as pressure, density and free energy. We will see how this is done in Sec. 2.3.2.

Generally, the partition function, denoted by Z, is defined as the trace of the operator $e^{-\beta H}$. Written in a basis of energy eigenfunctions, this is

$$Z = \operatorname{Tr} e^{-\beta H} = \sum_{n} \langle n | e^{-\beta H} | n \rangle = \sum_{n} e^{-\beta E_{n}}, \qquad (2.63)$$

where $\beta = 1/T$, and the sum is taken over all possible energy eigenstates $|n\rangle$ of the system. The trace is independent of the choice of basis, and for the single-particle system described in Sec. 2.2.1 it would be more convenient to use the position eigenstate basis:

$$Z = \operatorname{Tr} e^{-\beta H} = \int \mathrm{d}x \, \langle x | e^{-\beta H} | x \rangle.$$
(2.64)

Now compare Eq. (2.64) to Eq. (2.39). We see that if we perform a so called *Wick rotation* to imaginary time, $t = -i\tau$, and integrate τ from 0 to β , we can insert the latter directly into the expression for the partition function, yielding

$$Z = \int dx \int_{x(0)=x(\beta)=x} \mathcal{D}[x(\tau)] e^{i(-i)\int_{0}^{\beta} d\tau \left[\frac{m}{2} \left(\frac{1}{-i} \frac{dx}{d\tau}\right)^{2} - V(x(\tau))\right]}$$

$$= \int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)] e^{-\int_{0}^{\beta} d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau}\right)^{2} + V(x(\tau))\right]},$$

$$= \int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)] e^{-S_{E}[x(\tau)]}.$$
 (2.65)

The ordinary Minkowski metric, $ds^2 = dt^2 - d\mathbf{r}^2$, is now transformed to the Euclidean metric $ds^2 = d\tau^2 + d\mathbf{r}^2$. S_E is therefore called the Euclidean action, and is defined as

$$S_E \equiv \int_0^\beta \mathrm{d}\tau \ L_E,\tag{2.66}$$

where L_E is the Euclidean Lagrangian,

$$L_E \equiv -L(t \to -i\tau). \tag{2.67}$$

Interestingly, in this case we see that the Euclidean Lagrangian has the same functional form as the ordinary Hamiltonian.

We also note that because of the expression for the trace, Eq. (2.64), we only integrate over functions that are periodic in τ with a period of β .

Again, it is straightforward to extend the formalism to field theory. For a scalar field, we obtain

$$Z = \int_{\phi(\mathbf{x},0)=\phi(\mathbf{x},\beta)} \mathcal{D}[\phi] \ e^{-S_E[\phi]}, \tag{2.68}$$

where the integration runs over all periodic fields $\phi(\mathbf{x}, \tau)$ with a period of β in imaginary time. For fermions, we get

$$Z = \int_{\psi(\mathbf{x},0) = -\psi(\mathbf{x},\beta)} \mathcal{D}[\bar{\psi},\psi] \ e^{-S_E[\bar{\psi},\psi]},\tag{2.69}$$

and note that the fields $\psi(\mathbf{x}, \tau)$ now have to be *antiperiodic*¹. In both cases,

$$S_E = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \,\mathcal{L}_E,\tag{2.70}$$

where

$$\mathcal{L}_E = -\mathcal{L}(t \to -i\tau) \tag{2.71}$$

is the Euclidean Lagrangian density.

From this point onward we will drop the subscript E on the Euclidean action and just denote it S, as it is the only action we will use.

Eq. (2.63) is the *canonical* partition function. If our fields are in a medium of finite density with which they can interact and exchange particles, statistical mechanics tells us that we have to use the *grand canonical partition function* instead. This is usually defined as

$$Z = \operatorname{Tr} e^{-\beta(H-\mu_i \hat{N}_i)}, \qquad (2.72)$$

where μ_i are the chemical potentials and \hat{N}_i are the number operators for each type of particle in the system.

More generally, we can associate a chemical potential μ_i with *each conserved charge* in the system. These conserved charges can indeed be particle numbers, but they can also be electrical charge, isospin et cetera. Suppose we have a set of *independently* conserved charges Q_i ; the partition function is then

$$Z = \operatorname{Tr} e^{-\beta(H-\mu_i \hat{Q}_i)},\tag{2.73}$$

where \hat{Q}_i are the charge operators. The easiest way to incorporate these finite-density terms in field theory is to simply add them to the Hamiltonian density. We then write

$$\mathcal{H} = \mathcal{H}_0 - \mu_i \hat{\rho}_i, \tag{2.74}$$

where \mathcal{H}_0 is the zero-density Hamiltonian and the $\hat{\rho}_i$ are charge density operators. From the Hamiltonian we then calculate the Lagrangian.

Note that including chemical potentials breaks the Lorentz invariance of a theory. This can be shown by applying a Lorentz transformation to the fields and noting that the Lagrangian looks different after doing so, but it can also be understood quite intuitively: When a particle (e.g. a field), is in vacuum, all frames of reference are equivalent. There is no "preferred" speed, position or direction. If, on the other hand, the particle moves through a medium of finite density, its speed is measured relative to the medium; different velocities are no longer equivalent. The properties of the medium may differ from one place in spacetime to another; suddenly direction and position matters.

¹See section 2.5 of Ref. [31] for further details.

2.3.2 Thermodynamic quantities

Thermal averages

In the grand canonical ensemble, the *thermal average* of an operator \mathcal{O} is defined as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \operatorname{Tr} \left(\mathcal{O} e^{-\beta (H - \mu_i \hat{Q}_i)} \right).$$
 (2.75)

That the r.h.s. has to be normalised with the factor 1/Z is easy to see since, necessarily, $\langle 1 \rangle = 1$. This can of course also be written as a path integral,

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] \ \mathcal{O}e^{-S},$$
(2.76)

where ϕ is a generic field. Note the analogy with Eq. (2.49).

Thermal propagators

The thermal propagator of a scalar field theory is defined as

$$G(x-y) = \langle T\phi(x)\phi(y) \rangle, \qquad (2.77)$$

while for a fermionic field theory it is

$$G(x-y) = \langle T\psi(x)\bar{\psi}(y)\rangle.$$
(2.78)

Using Eq. (2.76) we can write Eq. (2.77) as

$$G(x-y) = \frac{1}{Z} \int \mathcal{D}[\phi] \ \phi(x)\phi(y) \ e^{-S}.$$
 (2.79)

Note that this is basically the same expression as Eq. (2.50). This means that the method we used to find vacuum propagators in Sec. 2.2.3 can also be used to find thermal propagators.

Thermodynamic potential

After all fields have been integrated out from the partition function, we are left with an expression of the form

$$Z = e^{-S_{\text{eff}}}.$$
(2.80)

The quantity S_{eff} is called the *effective action*. From statistical physics we have the relations

$$Z = e^{-\beta V\Omega} = e^{\beta VP}, \qquad (2.81)$$

where Ω is the thermodynamic potential (also called grand canonical potential or just grand potential) per unit volume², and P is the pressure. This gives

$$\Omega = -P = \frac{S_{\text{eff}}}{\beta V} = -\frac{1}{\beta V} \ln Z.$$
(2.82)

When the system is in a stable thermal equilibrium, Ω is at its global minimum [35] (with respect to the thermodynamic variables).

²In ordinary statistical physics the convention is usually $Z = e^{-\beta\Omega}$, so that Ω has the dimension of energy instead of energy density.

State variables

In a system with N independently conserved charges, each charge associated with a chemical potential μ_i (i = 1, 2, ..., N), the density of charges of type i is given by

$$\rho_i = \frac{\langle \hat{Q}_i \rangle}{V} = \frac{1}{VZ} \operatorname{Tr} \left(\hat{Q}_i e^{-\beta (H - \mu_j \hat{Q}_j)} \right) = \frac{1}{\beta VZ} \frac{\partial Z}{\partial \mu_i} = \frac{1}{\beta V} \frac{\partial}{\partial \mu_i} \ln Z = -\frac{\partial \Omega}{\partial \mu_i}.$$
 (2.83)

The average energy density, or internal energy density, is

$$\mathcal{U} = \frac{\langle U \rangle}{V} = \frac{1}{VZ} \operatorname{Tr} \left(H e^{-\beta (H - \mu_i \hat{Q}_i)} \right) = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln Z + \mu_i \rho_i = \Omega + TS + \mu_i \rho_i, \quad (2.84)$$

where

$$S = -\frac{\partial\Omega}{\partial T} \tag{2.85}$$

is the entropy density.

2.3.3 Frequency sums

As we will see in subsequent chapters, in thermal field theory one frequently has to evaluate sums of the form

$$\sigma = \frac{1}{2}T \sum_{n=-\infty}^{\infty} \ln(\omega_n^2 + \varepsilon^2), \qquad (2.86)$$

where ω_n are called *Matsubara frequencies* and are given by

$$\omega_n = \begin{cases} 2\pi nT & \text{for bosons} \\ (2n+1)\pi T & \text{for fermions.} \end{cases}$$
(2.87)

These frequencies are introduced in the Fourier expansions of the fields involved, to ensure the (anti)periodicity of these. (See Secs. 2.3.4 and 2.3.5 for details.) Following Ref. [36], we will now look at the most common method of evaluating these sums—the method of complex contour integration. We will first go through the calculations in detail for the bosonic case, and thereafter show what changes we have to make for the recipe to apply to fermions.

First, we rewrite the sum as follows:

$$\sigma_b = \frac{1}{2}T\sum_n \ln(-(i\omega_n)^2 + \varepsilon^2), \qquad (2.88)$$

then we differentiate with respect to ε ,

$$\frac{\mathrm{d}\sigma_b}{\mathrm{d}\varepsilon} = -T\varepsilon \sum_n \frac{1}{(i\omega_n)^2 - \varepsilon^2}.$$
(2.89)

By the Residue Theorem [37], we can express the sum over n as a contour integral over the complex variable ω ,

$$\frac{\mathrm{d}\sigma_b}{\mathrm{d}\varepsilon} = -T\varepsilon \sum_n \frac{\beta}{2} \operatorname{Res} \left\{ \frac{1}{\omega^2 - \varepsilon^2} \coth \frac{\beta\omega}{2}; \quad \omega = i\omega_n = i2\pi nT \right\}$$

$$= -\frac{\varepsilon}{2\pi i} \oint_C \frac{1}{2} \frac{1}{\omega^2 - \varepsilon^2} \coth \frac{\beta\omega}{2},$$
(2.90)

since the hyperbolic cotangent has poles at $\frac{1}{2}\beta\omega = i\pi n$, that is, at $\omega = i\omega_n$, and the residues there are $\frac{2}{\beta}$. The contour *C*, shown in Fig. 2.3a, is closed at infinity and encircles all of these poles.

The hyperbolic cotangent is bounded everywhere except on the imaginary axis, and the factor $1/(\omega^2 - \varepsilon^2)$ decreases fast enough that we may instead close the contour around the poles at



Figure 2.3: (a) The original contour C, encircling the poles of the hyperbolic cotangent. (b) The contours C^+ and C^- , enclosing the poles at $\pm \varepsilon$.

 $\omega = \pm \varepsilon$, without getting contributions from the semicircles. This is shown in Fig. 2.3b. Then,

$$\frac{\mathrm{d}\sigma_{b}}{\mathrm{d}\varepsilon} = -\frac{\varepsilon}{2\pi i} \oint_{(C^{+}\cup C^{-})} \frac{1}{\omega^{2} - \varepsilon^{2}} \frac{1}{2} \coth\frac{\beta\omega}{2}$$

$$= \varepsilon \sum_{\omega=\pm\varepsilon} \operatorname{Res} \left\{ \frac{1}{\omega^{2} - \varepsilon^{2}} \frac{1}{2} \coth\frac{\beta\omega}{2} \right\}$$

$$= \varepsilon \left(\frac{1}{2\varepsilon} \frac{1}{2} \coth\frac{\beta\varepsilon}{2} - \frac{1}{2\varepsilon} \frac{1}{2} \coth\frac{\beta(-\varepsilon)}{2} \right)$$

$$= \frac{1}{2} \coth\frac{\beta\varepsilon}{2}$$

$$= \frac{1}{2} + \frac{e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}}$$
(2.91)

Note that the sign is switched in the second line because of the orientation of the contours. It is also worth noting that the second term in the final expression is the well-known Bose-Einstein distribution function. Taking the antiderivative with respect to ε , we obtain

$$\sigma_b = \frac{1}{2}\varepsilon + T\ln(1 - e^{-\beta\varepsilon}) + \text{const}, \qquad (2.92)$$

where the constant is actually divergent, but independent of T and ε ; it can therefore be ignored.

The procedure for the fermionic case is identical, except that one has to use the fermionic Matsubara frequencies and replace the hyperbolic cotangent with the hyperbolic tangent. The final result is then

$$\sigma_f = \frac{1}{2}\varepsilon + T\ln(1 + e^{-\beta\varepsilon}) + \text{const.}$$
(2.93)

Differentiating this expression with respect to ε gives, as one may expect, the Fermi-Dirac distribution function.

2.3.4 The scalar field at finite temperature

We will now obtain an expression for the thermodynamic potential of a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2, \qquad (2.94)$$
where m is the mass.

In Euclidean space, the Lagrangian becomes

$$\mathcal{L}_E = \frac{1}{2} (\partial_\mu \phi) (\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2, \qquad (2.95)$$

giving the action

$$S = \int_0^\beta d\tau \int d^3x \, \left[\frac{1}{2} (\partial_\mu \phi) (\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 \right] = \frac{1}{2} \int_0^\beta d\tau \int d^3x \, \phi(-\partial^2 + m^2) \phi, \tag{2.96}$$

where we have integrated by parts and ignored a surface term.

It is easier to work in frequency-momentum space, so we write the field as a Fourier series:

$$\phi(\tau, \mathbf{x}) = \frac{1}{\sqrt{\beta V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} \phi_{n, \mathbf{p}} e^{i(\omega_n \tau + \mathbf{p} \cdot \mathbf{x})}, \qquad (2.97)$$

where the coefficients are given by

$$\phi_{n,\mathbf{p}} = \frac{1}{\sqrt{\beta V}} \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3 x \; \phi(\tau, \mathbf{x}) e^{-i(\omega_n \tau + \mathbf{p} \cdot \mathbf{x})}.$$
 (2.98)

Using the Matsubara frequencies $\omega_n = 2\pi nT$ in the exponent ensures the τ -periodicity of ϕ . Additionally, we have applied periodic boundary conditions in a finite volume V, resulting in discrete momenta. In the momentum representation the action becomes

$$S = \frac{1}{2} \sum_{n} \sum_{\mathbf{p}} \phi_{n,\mathbf{p}} (\omega_n^2 + \mathbf{p}^2 + m^2) \phi_{-n,-\mathbf{p}}.$$
 (2.99)

For the field to be real, it is easily shown from Eq. (2.97) that we must have $\phi_{-n,-\mathbf{p}} = \phi_{n,\mathbf{p}}$. The partition function is then given by the Gaussian integral

$$Z = \int \mathcal{D}[\phi] \ e^{-\frac{1}{2}\sum_{n}\sum_{\mathbf{p}}\phi_{n,\mathbf{p}}G_{n,\mathbf{p}}^{-1}\phi_{n,\mathbf{p}}}, \qquad (2.100)$$

where $G_{n,\mathbf{p}}^{-1} = (\omega_n^2 + \mathbf{p}^2 + m^2)$ is the inverse thermal propagator. Z can be calculated using Eq. (2.16), yielding

$$Z = e^{-\frac{1}{2}\operatorname{Tr}\,\ln G^{-1}},\tag{2.101}$$

where, since G^{-1} is diagonal, the trace is simply

$$\operatorname{Tr} \ln G^{-1} = \sum_{n} \sum_{\mathbf{p}} \ln(\omega_n^2 + \mathbf{p}^2 + m^2)$$
(2.102)

Taking the continuum limit $\sum_{\mathbf{p}} \to V \int \frac{\mathrm{d}^3 p}{(2\pi)^3}$ and using Eq. (2.82) we get the thermodynamic potential

$$\Omega = \frac{1}{2}T\sum_{n} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} \ln(\omega_{n}^{2} + \mathbf{p}^{2} + m^{2}).$$
(2.103)

We already know how to calculate the frequency sum; by using Eq. (2.92) we obtain

$$\Omega = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \frac{1}{2} E_{\mathbf{p}} + T \ln(1 - e^{-\beta E_{\mathbf{p}}}) \right\},\tag{2.104}$$

where we have defined $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2}$.

It should be noted that the transition to Fourier space is usually not done this explicitly. It is common to take the trace of the operator matrix directly, so that

$$Z = e^{-\frac{1}{2}\operatorname{Tr}\,\ln(-\partial^2 + m^2)}.$$
(2.105)

This is completely equivalent to Eq. (2.101), since the Fourier transformation just amounts to a change of basis, and the trace of a matrix is independent of the basis in which it is taken:

$$\operatorname{Tr} \ln(-\partial^2 + m^2) = \operatorname{Tr} \ln(\omega_n^2 + \mathbf{p}^2 + m^2).$$
(2.106)

The thermal propagator is found by inverting G^{-1} ; we obtain

$$G_n(\mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}.$$
(2.107)

2.3.5 The Dirac field at finite temperature and density

Next, we will find the grand canonical partition function and thermodynamic potential of a fermion field with Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi.$$
(2.108)

We have already derived the number density for fermions; it is the zeroth component of the current density, Eq. (1.27). Since it does not contain derivatives, there is no need to go via the Hamiltonian formalism. We just add the finite-density term μn directly to the Lagrangian, which may then be written

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m + \mu\gamma^{0})\psi.$$
(2.109)

The action is then

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3 x \, \bar{\psi} \left(\gamma^0 \frac{\partial}{\partial \tau} - i \gamma \cdot \nabla + m - \mu \gamma^0 \right) \psi. \tag{2.110}$$

Again, we use a frequency-momentum representation of the fields,

$$\psi(\tau, \mathbf{x}) = \frac{1}{\sqrt{\beta V}} \sum_{n, \mathbf{p}} \psi_{n, \mathbf{p}} e^{i(\omega_n \tau + \mathbf{p} \cdot \mathbf{x})}, \qquad (2.111)$$

with Fourier coefficients

$$\psi_{n,\mathbf{p}} = \frac{1}{\sqrt{\beta V}} \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3 x \; \psi(\tau, \mathbf{x}) e^{-i(\omega_n \tau + \mathbf{p} \cdot \mathbf{x})}.$$
 (2.112)

Here we use the fermionic Matsubara frequencies, $\omega_n = (2n+1)\pi T$, ensuring that ψ is antiperiodic in τ . We thus get the action

$$S = \sum_{n,\mathbf{p}} \bar{\psi}_{n,\mathbf{p}} \left[\gamma^0(i\omega_n) + \boldsymbol{\gamma} \cdot \mathbf{p} + m - \mu \gamma^0 \right] \psi_{n,\mathbf{p}}.$$
 (2.113)

We insert this into Eq. (2.69) and obtain

$$Z = \int \mathcal{D}[\bar{\psi}, \psi] \ e^{-\sum_{n,\mathbf{p}} \bar{\psi}_{n,\mathbf{p}} G_{n,\mathbf{p}}^{-1} \psi_{n,\mathbf{p}}}, \qquad (2.114)$$

where the inverse thermal propagator $G_{n,\mathbf{p}}^{-1} = \left[\gamma^0(i\omega_n) + \boldsymbol{\gamma} \cdot \mathbf{p} + m - \mu\gamma^0\right]$, is clearly a matrix in both Dirac and frequency-momentum space. This allows us to interpret Z as a Gaussian integral over Grassmann variables, and by using Eq. (2.20) we thus obtain

$$Z = e^{\text{Tr} \ln G^{-1}}.$$
 (2.115)

We first evaluate the frequency-momentum part of the trace:

Tr ln
$$G^{-1} = \sum_{n,\mathbf{p}} \operatorname{tr} \ln \left[\gamma^0(i\omega_n) + \boldsymbol{\gamma} \cdot \mathbf{p} + m - \mu \gamma^0 \right].$$
 (2.116)

The remainder is a quite ordinary matrix trace, and by applying the relation tr $(\ln M) = \ln(\det M)$ we get

$$\operatorname{Tr} \ln G^{-1} = \sum_{n,\mathbf{p}} \ln \det \left[\gamma^{0}(i\omega_{n}) + \boldsymbol{\gamma} \cdot \mathbf{p} + m - \mu \gamma^{0} \right] \\ = 2 \sum_{n,\mathbf{p}} \left[\ln(i\omega_{n} + \varepsilon_{-}) + \ln(i\omega_{n} - \varepsilon_{+}) \right], \qquad (2.117)$$

where we have defined $\varepsilon_{\pm} \equiv E_{\mathbf{p}} \pm \mu = \sqrt{\mathbf{p}^2 + m^2} \pm \mu$. Since we sum over both positive and negative values of n, the logarithmic terms can be symmetrised:

$$2\sum_{n=-\infty}^{\infty} \ln(i\omega_n + \varepsilon_-) = \sum_{n=-\infty}^{\infty} \ln(i\omega_n + \varepsilon_-) + \sum_{n=-\infty}^{\infty} \ln(-i\omega_n + \varepsilon_-)$$
$$= \sum_{n=-\infty}^{\infty} \ln(\omega_n^2 + \varepsilon_-^2).$$
(2.118)

Likewise, $2\sum_n \ln(i\omega_n - \varepsilon_+) = \sum_n \ln(\omega_n^2 + \varepsilon_+^2)$. Therefore,

$$\Omega = -T \sum_{n} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} \left\{ \ln(\omega_{n}^{2} + \varepsilon_{-}^{2}) + \ln(\omega_{n}^{2} + \varepsilon_{+}^{2}) \right\}, \qquad (2.119)$$

and by using Eq. (2.93), we obtain

$$\Omega = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ E_{\mathbf{p}} + T \ln[1 + e^{-\beta(E_{\mathbf{p}} - \mu)}] + T \ln[1 + e^{-\beta(E_{\mathbf{p}} + \mu)}] \right\}.$$
 (2.120)

Note that there are now two finite-temperature terms; the second is the contribution from the antiparticles. We can now use Eq. (2.83) to find the charge density, which is

$$\rho = -\frac{\partial\Omega}{\partial\mu} = 2\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \frac{1}{e^{\beta(E_{\mathbf{p}}-\mu)}+1} - \frac{1}{e^{\beta(E_{\mathbf{p}}+\mu)}+1} \right\}.$$
(2.121)

It is now clear how the charge density depends on the chemical potential: When $\mu > 0$, there are more particles than antiparticles in the system, and $\rho > 0$. Likewise, when $\mu < 0$, $\rho < 0$. Finally, when $\mu = 0$, the numbers of particles and antiparticles are the same, and the charge is zero.

In the limit $\mu = 0$ we also see that

$$\Omega = -4 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \frac{1}{2} E_{\mathbf{p}} + T \ln(1 + e^{-\beta E_{\mathbf{p}}}) \right\},\tag{2.122}$$

which we should compare with Eq. (2.104). Note the factor 4, which stems from the fact that Dirac spinors have four degrees of freedom—spin up, spin down, particle and antiparticle—whereas a real scalar field has only one. In this limit, the thermal propagator of the theory is given by

$$G_n(\mathbf{p}) = \frac{1}{\gamma^0(i\omega_n) + \boldsymbol{\gamma} \cdot \mathbf{p} + m} = \frac{m - \gamma^0(i\omega_n) - \boldsymbol{\gamma} \cdot \mathbf{p}}{\omega_n^2 + \mathbf{p}^2 + m^2}.$$
(2.123)

2.4 Phase structure

The remainder of this thesis will largely be concerned with analysing the phase structure of quark matter. Before doing this we need to clarify what we mean by a *phase*, and we should be familiar with the different types of phase transitions. Subsequently we will see how to use these definitions to create *phase diagrams*.

2.4.1 Phases

Generally speaking, we say that two distinct states of a system are in different phases if there is an abrupt change in their physical properties while transforming from one to the other. Temporarily accepting this rather sloppy definition, we can mention one very common example: Water. Water has three well-known phases, namely a solid phase (ice), a liquid phase (water) and a vapourous phase (steam). These can be depicted in a phase diagram, as shown in Fig. 2.4. Indeed, this figure is a good illustration of the phase diagram of most pure substances.

To create such a diagram, we have to select one or more *order parameters*. An order parameter is a quantity which vanishes in one phase and has a nonzero value in another, and can therefore be used to distinguish between the phases. In the case of water, the order parameters are based



Figure 2.4: Phase diagram of a typical pure substance. Note the points T and C; T is the *triple point*, where all three phases can coexist simultaneously, while C is a *critical point*, beyond which the liquid and vapour phases are indistinguishable.

on density differences. For instance, for the transition between the solid and liquid phases we use $\delta \rho = \rho - \rho_s$ as the order parameter, where ρ_s is the density in the solid phase [32].

For our purposes, the above definition of a phase does not suffice. We will need a stricter one, namely the following:

Two different states of a system are in different phases if their symmetries differ.

With this definition it is easy to select order parameters. We just have to find which symmetries are spontaneously broken and identify the quantities that break them; these will be our order parameters.

2.4.2 Phase transitions

Without going in too much depth on the subject, we will now describe a scheme for identifying three different ways of going from one phase to another:

- A *first order phase transition* is one in which the order parameter jumps discontinuously from zero to a finite value.
- At a *second order phase transition*, the order parameter is continuous, but its derivative is not.
- At a *cross-over* the order parameter changes rapidly in a narrow range without vanishing, which means that no derivative has a discontinuity.

The behaviour of the order parameter is for each case shown graphically in Fig. 2.5.

2.4.3 Phase diagrams

The principle behind creating phase diagrams is simple. First one has to decide on suitable axes for the diagram. In the example with water we used temperature and pressure; in subsequent chapters we will use other combinations of parameters, such as temperature and chemical potential. To generalise, we will denote the axis variables q_1 and q_2 .

By close inspection of the theory, one next identifies the spontaneously broken symmetries and the quantities that break them. These are our order parameters, and we denote them here as



Figure 2.5: Typical behaviour of the order parameter near a phase transition: (a) First order phase transition. (b) Second order phase transition. (c) Cross-over.

 p_1, p_2, \ldots, p_N . One then calculates the thermodynamic potential, $\Omega(q_1, q_2, p_1, p_2, \ldots, p_N)$, as this contains all the thermodynamics of the system.

Thereafter one minimises the potential with respect to the order parameters for each point in the diagram, thus finding the equilibrium values of these. This amounts to simultaneously solving the set of equations

$$\begin{pmatrix} \frac{\partial \Omega}{\partial p_1} \end{pmatrix}_{q_1,q_2} = 0,$$

$$\begin{pmatrix} \frac{\partial \Omega}{\partial p_2} \end{pmatrix}_{q_1,q_2} = 0,$$

$$\vdots$$

$$\begin{pmatrix} \frac{\partial \Omega}{\partial p_N} \end{pmatrix}_{q_1,q_2} = 0.$$

These are often referred to as gap equations.

Finally, one has to locate the phase transitions—the lines where two or more phases meet—and identify their type. It is common to draw first-order transitions as solid lines and second-order transitions as dashed lines.

Chapter 3

The O(N) linear sigma model

In this chapter we study the O(N) linear sigma model to leading order in the 1/N expansion at finite temperature and finite isospin chemical potential, for both zero and nonzero vacuum masses of the pions. We find that pion condensation occurs at low temperatures for chemical potentials larger than the pion mass, and that the transition to this phase is second order.

3.1 Introduction

The O(N) linear sigma model is a theory of N real scalar fields ϕ_i , with O(N)-symmetric quartic self-interactions. The Lagrangian of the theory is typically written as

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \frac{1}{2} m_0^2 \phi_i \phi_i - \frac{\lambda_0}{8N} (\phi_i \phi_i)^2, \qquad (3.1)$$

where m_0 is a bare mass parameter and λ_0 is the bare coupling constant. Comparing to Eq. (1.51), we see that the coupling constant has been scaled down by a factor of 1/N. This is to ensure that the relative strengths of the couplings between fields is not changed when N is varied [38].

In Sec. 1.3.2, we showed that the O(N) symmetry of the model is spontaneously broken down to O(N-1) in the ground state. Furthermore, we saw that this causes the appearance of N-1 massless Goldstone bosons. If one considers only two massless quark flavours, the QCD Lagrangian exhibits an $SU(2)_L \times SU(2)_R$ flavour symmetry which is spontaneously broken down to $SU(2)_V$ at low temperatures and densities. From group theory we learn that $SU(2) \times SU(2)$ is locally isomorphic¹ to O(4), while SU(2) is isomorphic to O(3) [27]. Therefore, when N = 4, the linear sigma model has the same symmetry breaking pattern and critical behaviour as QCD. The N-1 = 3 Goldstone bosons are then identified with the three pions, and the remaining massive field represents the sigma meson. Because of this, the linear sigma model in 3 + 1 dimensions can be used as a low-energy effective theory for QCD.

We will use the *one-particle irreducible* (1PI) formalism to calculate the thermodynamic potential. In this formalism one expands around the true vacuum, where nonzero expectation values for the fields are allowed. This is in contrast with ordinary perturbation theory, in which one expands around the trivial vacuum, where all vacuum expectation values are zero [27]. In addition to the 1PI formalism, the 2PI formalism [39] is commonly used, as well as other *n*PI formalisms.

One method of obtaining an approximation to the thermodynamic potential of the model is by making a perturbative expansion in powers of the coupling constant. This, however, has the major drawback that it can only be done for small couplings. We will therefore expand in powers of 1/N, and in the end we will only keep leading-order terms. If the number of fields is four we then expect to be making an error of about 25%.

The O(N) sigma models have been studied in the 1PI formalism with the 1/N expansion before, at both zero [38, 40] and finite [41, 42, 43] temperature. Other approaches have been used as well, among them optimised perturbation theory [44] and the aforementioned 2PI formalism [45, 46, 47, 48, 49, 50, 51, 52, 53, 54] (for a review, see Ref. [55]). In the present chapter our goal is

¹When two groups are isomorphic they are, group theoretically, indistinguishable.

to study pion condensation and we therefore include a finite isospin chemical potential. This has been done very recently using the 2PI approach by Andersen [56]. To leading order the 1PI and 2PI approaches are equivalent, so we expect to get the same results here.

3.2 Lagrangian density

In the following we take N to be an even number. The Lagrangian of a Bose gas of N/2 massive charged scalar fields can be written as

$$\mathcal{L} = (\partial_{\mu}\Phi_a^*)(\partial^{\mu}\Phi_a) - \frac{\lambda_0}{2N} \left(\Phi_a^*\Phi_a - \frac{N}{2}f_0^2\right)^2, \qquad (3.2)$$

where f_0 is a constant defined below, $a = 1, 2, \ldots, N/2$ and

$$\Phi_a = \frac{1}{\sqrt{2}}(\phi_{2a-1} + i\phi_{2a}). \tag{3.3}$$

The current density is given by Eq. (1.23):

$$j_a^{\mu} = i \left(\Phi_a \partial^{\mu} \Phi_a^* - \Phi_a^* \partial^{\mu} \Phi_a \right).$$
(3.4)

Written in terms of the real scalar fields ϕ_i the Lagrangian is

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \frac{\lambda_0}{8N} (\phi_i \phi_i - N f_0^2)^2.$$
(3.5)

(Note that i = 1, 2, ..., N.) By expanding the square term it is easily shown that this expression is equivalent to Eq. (3.1), except for an irrelevant constant. It is then also seen that f_0 is related to the bare mass and the coupling constant by

$$f_0^2 = -\frac{2m_0^2}{\lambda_0},\tag{3.6}$$

and we assume that $f_0 > 0$. To mimic the explicit breaking of chiral symmetry in QCD by finite quark masses, which is the cause of the observed finite masses of the pions, we introduce a phenomenological symmetry breaking term in the Lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \frac{\lambda_0}{8N} (\phi_i \phi_i - N f_0^2)^2 + \sqrt{N} H \phi_N.$$
(3.7)

H is a constant, and we assume $H \ge 0$. When H = 0 the Lagrangian has an O(N) symmetry, corresponding to rotations among the *N* real fields ϕ_i . When H > 0 this symmetry is explicitly broken down to O(N-1). In terms of the fields ϕ_i , the current density (3.4) is now written

$$j_a^{\mu} = \phi_{2a-1}\partial_{\mu}\phi_{2a} - \phi_{2a}\partial_{\mu}\phi_{2a-1}.$$
(3.8)

For N = 4, we let the fields Φ_1 and Φ_1^* (that is, $\phi_1 \pm i\phi_2$) correspond to the two charged pions, π^+ and π^- , respectively, while we let the fields ϕ_3 and ϕ_4 represent the neutral pion and the sigma particle. Generally (for any N), we call the the fields $\phi_1, \ldots, \phi_{N-1}$ pions.

3.3 Tree-level parameters

For simplicity, we now write the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \mathcal{V}(\phi_i), \qquad (3.9)$$

where $\mathcal{V}(\phi_i)$ is the classical potential

$$\mathcal{V}(\phi_i) = \frac{\lambda_0}{8N} (\phi_i \phi_i - N f_0^2)^2 - \sqrt{N} H \phi_N.$$
(3.10)

To find the (classical) vacuum state we have to locate the global minima of $\mathcal{V}(\phi_i)$. For the case H = 0, this is exactly what we did in Sec. 1.3.2, where we, because of the O(N) symmetry, found an infinite number of equivalent ground states. When H > 0, the symmetry is explicitly broken, and the potential is lowered in the Nth direction. This means that there is only one vacuum state, and that in this state only the field ϕ_N acquires a vacuum expectation value. We therefore write $\phi_i = (0, 0, \ldots, 0, \sqrt{N\sigma})$, and obtain the following expression for the potential:

$$\mathcal{V}(\sigma) = \frac{\lambda_0 N}{8} (\sigma^2 - f_0^2)^2 - NH\sigma.$$
(3.11)

We then minimise the potential by solving the equation

$$0 = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\sigma} = \frac{\lambda_0 N}{2} (\sigma - f_0^2) \sigma - NH. \tag{3.12}$$

The solution is [50]

$$\sigma = \frac{f_{\pi}}{\sqrt{N}} \equiv \frac{2}{\sqrt{3}} f_0 \cos \frac{\theta}{3},\tag{3.13}$$

where

$$\theta = \arccos\left(\frac{\sqrt{27}H}{\lambda_0 f_0^3}\right). \tag{3.14}$$

The constant f_{π} is the *pion decay constant*, which for N = 4 is a quantity determined by experiment. Since we are free to choose coordinates so that the ground state points in the Nth direction also for H = 0, the above is completely general. In this case, we see that $\sigma = f_{\pi}/\sqrt{N} = f_0$.

By expanding the field ϕ_N around its vacuum expectation value, $\phi_N \to \sqrt{N}\sigma + \phi_N$, we obtain

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_i) (\partial^{\mu} \phi_i) - \frac{\lambda_0}{8N} \left(\phi_i \phi_i + N\sigma^2 + 2\sqrt{N}\sigma\phi_N - Nf_0^2 \right)^2 + \sqrt{N}H(\sqrt{N}\sigma + \phi_N).$$
(3.15)

Recall from Sec. 2.2.3 that the tree-level propagators of the theory are given by the quadratic terms in the Lagrangian—that is, terms of the form $\phi_i D_{ij}^{-1} \phi_j$. In this case D_{ij}^{-1} is diagonal, so we can read off the inverse tree-level pion and sigma propagators directly:

$$D_{\pi}^{-1} = D_{ii}^{-1} = -\partial^2 - m_{\pi}^2, \quad i \neq N, \text{ no sum over } i$$
 (3.16)

$$D_{\sigma}^{-1} = D_{NN}^{-1} = -\partial^2 - m_{\sigma}^2, \qquad (3.17)$$

The pion and sigma masses are given by

$$m_{\pi}^2 = \frac{\lambda_0}{2} (\sigma^2 - f_0^2), \qquad (3.18)$$

$$m_{\sigma}^2 = \frac{\lambda_0}{2} (3\sigma^2 - f_0^2). \tag{3.19}$$

and we immediately see that the pions are massless if H = 0.

Using that $\sigma = f_{\pi}/\sqrt{N}$, we can now express the parameters λ_0 and f_0 in terms of measurable quantities:

$$\lambda_0 = \frac{N(m_{\sigma}^2 - m_{\pi}^2)}{f_{\pi}^2}, \qquad (3.20)$$

$$f_0^2 = \frac{m_\sigma^2 - 3m_\pi^2}{N(m_\sigma^2 - m_\pi^2)} f_\pi^2.$$
(3.21)

By inserting these back into Eq. (3.12), one also finds that

$$H = \frac{m_\pi^2 f_\pi}{\sqrt{N}}.\tag{3.22}$$

The reason for prefixing the explicit symmetry breaking term with a factor of \sqrt{N} in Eq. (3.7) is now apparent; since $f_{\pi} \propto \sqrt{N}$, *H* does not depend on *N* at all.

3.4 Finite density

Each generator of a symmetry group gives rise to a conserved charge, each associated with a chemical potential [57]. The quantum state of the system can be specified by the eigenvalues of a complete set of commuting charges. For O(N) there are N(N-1)/2 charges, each corresponding to a conserved current,

$$j_{ij}^{\mu} = \phi_i \partial^{\mu} \phi_j - \phi_j \partial^{\mu} \phi_i, \qquad i \neq j.$$
(3.23)

Only N/2 of these charges commute, so we can only construct N/2 independently conserved currents; Eq. (3.8) is an example of this. In the following, however, we will only consider one conserved charge, and will therefore limit ourselves to introducing a single chemical potential. We will couple it to the current j_1^{μ} , which involves the fields ϕ_1 and ϕ_2 . In this way we can emulate the isospin chemical potential in QCD, and we therefore denote it by μ_I . We will now see how the incorporation of a chemical potential in the theory changes the classical potential.

The canonical conjugate momenta of the fields ϕ_i are

$$\pi_i = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_i)} = \partial^0 \phi_i = \partial_0 \phi_i, \qquad (3.24)$$

from which we can calculate the Hamiltonian density

$$\mathcal{H} = \pi_i \partial_0 \phi_i - \mathcal{L}$$

= $\frac{1}{2} \pi_i \pi_i - \frac{1}{2} (\partial_j \phi_i) (\partial^j \phi_i) + \mathcal{V}(\phi_i).$ (3.25)

From Sec. 2.3.1 we already know how to incorporate a chemical potential; we simply let $\mathcal{H} \to \mathcal{H} - \mu_I j_1^0$, where the zeroth component of the current is now written in terms of the canonical momenta:

$$j_1^0 = \phi_1 \pi_2 - \phi_2 \pi_1. \tag{3.26}$$

The Hamiltonian is then

$$\mathcal{H} = \frac{1}{2}\pi_i \pi_i - \frac{1}{2}(\partial_j \phi_i)(\partial^j \phi_i) + \mathcal{V}(\phi_j - \mu_I(\phi_1 \pi_2 - \phi_2 \pi_1)$$
(3.27)

The chemical potential leads to a shift in the canonical momenta π_1 and π_2 , which can be calculated using the Hamiltonian equations of motion. For instance,

$$\partial_0 \phi_1 = \frac{\partial \mathcal{H}}{\partial \pi_1} = \pi_1 + \mu_I \phi_2, \qquad (3.28)$$

and we thus obtain

$$\pi_1 = \partial_0 \phi_1 - \mu_I \phi_2, \tag{3.29}$$

$$\pi_2 = \partial_0 \phi_2 + \mu_I \phi_1, \tag{3.30}$$

$$\pi_i = \partial_0 \phi_i, \qquad i \ge 3. \tag{3.31}$$

Returning to the Lagrangian again, we calculate

$$\mathcal{L} = \pi_i \partial_0 \phi_i - \mathcal{H} = \frac{1}{2} (\partial_\mu \phi_i) (\partial^\mu \phi_i) - \mathcal{V}(\phi_i) + \mu_I (\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) + \frac{1}{2} \mu_I^2 (\phi_1^2 + \phi_2^2).$$
(3.32)

In the limit H = 0 we see that the original O(N) symmetry has been explicitly broken down to $O(2) \times O(N-2)$ by the chemical potential, where the O(2) symmetry rotates the fields ϕ_1 and ϕ_2 amongst themselves.

For future reference, it is worth noting that we would have arrived at the very same expression for \mathcal{L} by making the substitution $\partial_0 \Phi_1 \rightarrow (\partial_0 + i\mu_I)\Phi_1$ (and, of course, $\partial_0 \Phi_1^* \rightarrow (\partial_0 - i\mu_I)\Phi_1^*$) in the original Lagrangian, Eq. (3.2). The easiest way of incorporating multiple chemical potentials $\mu_1, \mu_2, \ldots, \mu_{N/2}$ is therefore to make the substitutions

$$\partial_0 \Phi_a \to (\partial_0 + i\mu_a) \Phi_a,$$
 (3.33)

where no sum over a is implied.

3.5 Finite temperature

As is usual in thermal field theory, we now go to a Euclidean spacetime and write the Lagrangian in Eq. (3.32) as

$$\mathcal{L}_E = \frac{1}{2} (\partial_\mu \phi_i) (\partial_\mu \phi_i) + \mathcal{V}(\phi_i) - \mu_I (\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) - \frac{1}{2} \mu_I^2 (\phi_1^2 + \phi_2^2).$$
(3.34)

Our next step will be to eliminate the quartic interaction term. To this end we introduce an auxiliary field α , and add the term

$$\mathcal{L}_{\alpha} = \frac{N}{2\lambda_0} \alpha^2 \tag{3.35}$$

to the Lagrangian. \mathcal{L}_{α} does not contain a kinetic term, so classically the α field has no dynamics, does not propagate and therefore does not alter the physics of the model in any way. The original partition function is recovered by integrating over α . (This will also yield an infinite constant factor, which is independent of physical parameters and can therefore safely be ignored.) Since a field is nothing but an integration variable in the path integral, it is always possible to shift it by a constant. If the α field is shifted according to

$$\alpha \to \alpha - \frac{i\lambda_0}{2N} (\phi_i \phi_i - N f_0^2), \qquad (3.36)$$

so that

$$\mathcal{L}_{\alpha} \to \frac{N}{2\lambda_0} \left[\alpha - \frac{i\lambda_0}{2N} (\phi_i \phi_i - N f_0^2) \right]^2, \qquad (3.37)$$

the quartic terms in the Lagrangian disappear:

$$\mathcal{L}_{E} \rightarrow \mathcal{L}_{E} + \mathcal{L}_{\alpha} = \frac{1}{2} (\partial_{\mu} \phi_{i}) (\partial_{\mu} \phi_{i}) - \frac{1}{2} i \alpha (\phi_{i} \phi_{i} - N f_{0}^{2}) - \sqrt{N} H \phi_{N} - \mu_{I} (\phi_{1} \partial_{0} \phi_{2} - \phi_{2} \partial_{0} \phi_{1}) - \frac{1}{2} \mu_{I}^{2} (\phi_{1}^{2} + \phi_{2}^{2}) + \frac{N}{2\lambda_{0}} \alpha^{2}.$$
 (3.38)

As an extra check we solve the Euler-Lagrange equation for the field α ,

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \alpha)} = \frac{N}{\lambda_0} \alpha - \frac{i}{2} (\phi_i \phi_i - N f_0^2) \quad \iff \quad \alpha = \frac{i \lambda_0}{2N} (\phi_i \phi_i - N f_0^2), \tag{3.39}$$

and insert it back into Eq. (3.38). We then recover Eq. (3.32), again proving that the inclusion of the auxiliary field has no effect on the physical content of the theory.

The partition function of the model is now given by the path integral

$$Z = \int \mathcal{D}[\phi] \mathcal{D}[\alpha] \ e^{-S}, \qquad (3.40)$$

Using integration by parts and assuming that the fields decrease fast enough as $x \to \infty$ that the surface terms vanish, we can make this integral quadratic in the neutral pion fields $\phi_3, \phi_4, \ldots, \phi_{N-1}$:

$$S = \frac{1}{2} \int_{0}^{\beta} d\tau \int d^{3}x \left\{ \phi_{i}(-\partial^{2} - i\alpha)\phi_{i} \right\} \\ + \int_{0}^{\beta} d\tau \int d^{3}x \left\{ \frac{1}{2} (\partial_{\mu}\phi_{j})(\partial_{\mu}\phi_{j}) - \frac{1}{2}i\alpha(\phi_{j}\phi_{j} - Nf_{0}^{2}) - \sqrt{N}H\phi_{N} - \mu_{I}(\phi_{1}\partial_{0}\phi_{2} - \phi_{2}\partial_{0}\phi_{1}) - \frac{1}{2}\mu_{I}^{2}(\phi_{1}^{2} + \phi_{2}^{2}) + \frac{N}{2\lambda_{0}}\alpha^{2} \right\}.$$
(3.41)

Here i = 3, 4, ..., N - 1 while j = 1, 2, N. These fields can now be integrated out to give

$$S' = \frac{1}{2}(N-3)\operatorname{Tr}\ln(-\partial^{2} - i\alpha) + \int_{0}^{\beta} \mathrm{d}\tau \int \mathrm{d}^{3}x \left\{ \frac{1}{2}(\partial_{\mu}\phi_{j})(\partial_{\mu}\phi_{j}) - \frac{1}{2}i\alpha(\phi_{j}\phi_{j} - Nf_{0}^{2}) - \sqrt{N}H\phi_{N} - \mu_{I}(\phi_{1}\partial_{0}\phi_{2} - \phi_{2}\partial_{0}\phi_{1}) - \frac{1}{2}\mu_{I}^{2}(\phi_{1}^{2} + \phi_{2}^{2}) + \frac{N}{2\lambda_{0}}\alpha^{2} \right\}.$$
 (3.42)

As before, there will be a nonzero vacuum expectation value for the fields $\phi_3, \phi_4, \ldots, \phi_N$, fixed in the ϕ_N direction by the explicit symmetry breaking term H. But now that we have included the chemical potential, either ϕ_1, ϕ_2 (or a linear combination of the two) will also acquire a vacuum expectation value. We use the O(2) symmetry to choose coordinates so that only $\langle \phi_1 \rangle \neq 0$. Finally, the α field will have a purely imaginary expectation value. We can show this by noting that $S(\alpha)^* = S(-\alpha)$ and changing the integration variable from α to $-\alpha$ in the following:

$$\langle \alpha \rangle^* = \int \mathcal{D}[\phi] \mathcal{D}[\alpha] \ \alpha \ e^{-S(-\alpha)} = -\int \mathcal{D}[\phi] \mathcal{D}[\alpha] \ \alpha \ e^{-S(\alpha)} = -\langle \alpha \rangle. \tag{3.43}$$

We therefore expand these three fields around their expectation values in the following manner:

$$\phi_1 \quad \to \quad \sqrt{N\rho + \phi_1},\tag{3.44}$$

$$\phi_N \rightarrow \sqrt{N\sigma} + \phi_N,$$
 (3.45)

$$\alpha \rightarrow iM^2 + \frac{\alpha}{\sqrt{N}}.$$
 (3.46)

The reason for including a factor of $1/\sqrt{N}$ in Eq. (3.46) will be made clear below. The logarithm in Eq. (3.42) can then be expanded as

$$\ln\left(-\partial^2 + M^2 - \frac{i\alpha}{\sqrt{N}}\right) = \ln(-\partial^2 + M^2) - \frac{1}{\sqrt{N}}\frac{i\alpha}{-\partial^2 + M^2} + \frac{1}{2N}\frac{\alpha^2}{(-\partial^2 + M^2)^2} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right), \quad (3.47)$$

and we can write the action in the following form:

$$S' = NS_N + \sqrt{N}S_{\sqrt{N}} + S_1 + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right), \qquad (3.48)$$

where

$$S_N = \frac{1}{2} \operatorname{Tr} \ln(-\partial^2 + M^2) + \beta V \left[\frac{1}{2} M^2 (\rho^2 + \sigma^2 - f_0^2) - H\sigma - \frac{1}{2} \mu_I^2 \rho^2 - \frac{M^4}{2\lambda_0} \right], \quad (3.49)$$

$$S_{\sqrt{N}} = \int_{0}^{\beta} d\tau \int d^{3}x \left\{ M^{2}(\rho\phi_{1} + \sigma\phi_{N}) - \frac{1}{2}i\alpha(\rho^{2} + \sigma^{2} - f_{0}^{2}) - H\phi_{N} - \mu_{I}\rho\partial_{0}\phi_{2} - \mu_{I}^{2}\rho\phi_{1} + \frac{M^{2}}{\lambda_{0}}i\alpha \right\} - \frac{1}{2}\mathrm{Tr}\frac{i\alpha}{-\partial^{2} + M^{2}}, \qquad (3.50)$$
$$S_{1} = -\frac{3}{2}\mathrm{Tr}\ln(-\partial^{2} + M^{2}) + \int_{0}^{\beta} d\tau \int d^{3}x \left\{ \frac{1}{2}(\partial_{\mu}\phi_{i})(\partial_{\mu}\phi_{i}) + \frac{1}{2}M^{2}\phi_{i}\phi_{i} \right\}$$

$$= -\frac{3}{2} \operatorname{Tr} \ln(-\partial^2 + M^2) + \int_0^{\tau} d\tau \int d^3x \left\{ \frac{1}{2} (\partial_{\mu} \phi_j) (\partial_{\mu} \phi_j) + \frac{1}{2} M^2 \phi_j \phi_j - \mu_I (\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) - \frac{1}{2} \mu_I^2 (\phi_1^2 + \phi_2^2) + \frac{1}{2\lambda_0} \alpha^2 \right\} + \frac{1}{4} \operatorname{Tr} \frac{\alpha^2}{(-\partial^2 + M^2)^2} (3.51)$$

Because of the definition of the fluctuations in α in Eq. (3.46), interaction terms of the form $\alpha \phi_i \phi_i$ are proportional to $1/\sqrt{N}$. This is convenient, because now one can see directly from the action that higher-order loop diagrams are suppressed. Had we made the more intuitive choice $\alpha \to iM^2 + \alpha$, a factor of N would appear in the term quadratic in α , thus the propagator of the α field would be suppressed by a factor of 1/N and the conclusion would be the same.

In thermal equilibrium, that is, at an extremum of the thermodynamic potential, the terms linear in the fluctuations vanish. This can be seen by making a Taylor expansion of the potential around its minimum. All terms of this form, and only terms of this form, are contained in $S_{\sqrt{N}}$, which therefore also vanishes. We thus obtain the partition function

$$Z = e^{-NS_N} \int \mathcal{D}[\phi] \mathcal{D}[\alpha] \ e^{-S_1} + \cdots .$$
(3.52)

3.6 Gap equations

To leading order in 1/N, the thermodynamic potential is $\Omega_N = S_N/\beta V$. Using the methods of Sec. 2.3.4 to calculate the trace, we obtain

$$\Omega_N = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \frac{1}{2} E_{\mathbf{p}} + T \ln(1 - e^{-\beta E_{\mathbf{p}}}) \right\} + \frac{1}{2} M^2 (\rho^2 + \sigma^2 - f_0^2) - H\sigma - \frac{1}{2} \mu_I^2 \rho^2 - \frac{M^4}{2\lambda_0}, \qquad (3.53)$$

where $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + M^2}$. This yields the following gap equations:

$$0 = \frac{\partial \Omega_N}{\partial \rho} = (M^2 - \mu_I^2)\rho, \qquad (3.54)$$

$$0 = \frac{\partial \Omega_N}{\partial \sigma} = M^2 \sigma - H, \qquad (3.55)$$

$$0 = \frac{\partial \Omega_N}{\partial (M^2)} = \frac{1}{2}Q(M,T) + \frac{1}{2}(\rho^2 + \sigma^2 - f_0^2) - \frac{M^2}{\lambda_0}, \qquad (3.56)$$

where we have defined

$$Q(M,T) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{1}{E_{\mathbf{p}}} \left(\frac{1}{2} + \frac{1}{e^{\beta E_{\mathbf{p}}} - 1} \right). \tag{3.57}$$

Since ρ is the expectation value of the charged pion fields, we call it a *charged pion condensate*. The above equations have several solutions, but we are interested in studying the phase where there is a nonvanishing pion condensate. This effectively limits the number of solutions to one; Eq. (3.54) then says that $M^2 = \mu_I^2$, and Eq. (3.55) gives the relation

$$\sigma = \frac{H}{\mu_I^2} \tag{3.58}$$

for the expectation value σ . Finally, we can rearrange Eq. (3.56) to yield an expression for ρ in terms of known parameters:

$$\rho^2 = f_0^2 - \frac{H^2}{\mu_I^4} + \frac{2\mu_I^2}{\lambda_0} - Q(\mu_I, T).$$
(3.59)

Note that Eqs. (3.54)-(3.56) are the same gap equations that were found in Ref. [56] (Eqs. 25–27), showing that the 1PI and 2PI approaches are equivalent to leading order in the 1/N expansion. Therefore, although the methods of renormalisation will differ, we expect to get the same results.

3.7 Mass corrections

In Sec. 3.3 we found the tree-level masses of the pions and the sigma particle expressed in terms of the parameters λ_0 and f_0 —Eqs. (3.18) and (3.19). We now wish to calculate the one-loop quantum contribution to these masses (Fig. 3.1), to leading order in 1/N. We then write

$$M_{\pi}^{2} = \frac{\lambda_{0}}{2}(\sigma^{2} - f_{0}^{2}) + \delta m_{\pi}^{2}, \qquad (3.60)$$

$$M_{\sigma}^{2} = \frac{\lambda_{0}}{2} (3\sigma^{2} - f_{0}^{2}) + \delta m_{\sigma}^{2}, \qquad (3.61)$$

where the one-loop corrections are given by

$$\delta m_{\pi}^{2} = \frac{\lambda_{0}}{8N} \left[4(N+1)T \sum_{n} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} G_{n}^{\pi}(\mathbf{p}) + 4T \sum_{n} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} G_{n}^{\sigma}(\mathbf{p}) \right], \qquad (3.62)$$

$$\delta m_{\sigma}^{2} = \frac{\lambda_{0}}{8N} \left[4(N-1)T \sum_{n} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} G_{n}^{\pi}(\mathbf{p}) + 12T \sum_{n} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} G_{n}^{\sigma}(\mathbf{p}) \right].$$
(3.63)



Figure 3.1: One-loop contribution to particle masses.

The factor $\lambda_0/8N$ comes from the vertex, while G^{π} and G^{σ} are the thermal propagators of the pions and the sigma particle that run in the loop. These are given by Eq. (2.107) with the appropriate masses inserted. To leading order, $\delta m_{\pi}^2 = \delta m_{\sigma}^2 = \delta m^2$, where

$$\delta m^2 = \frac{\lambda_0}{2} T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \ G_n^{\pi}(\mathbf{p}) = \frac{\lambda_0}{2} Q(M_{\pi}, T).$$
(3.64)

We then obtain the "dressed" masses

$$M_{\pi}^{2} = \frac{\lambda_{0}}{2} [\sigma^{2} - f_{0}^{2} + Q(M_{\pi}, T)], \qquad (3.65)$$

$$M_{\sigma}^{2} = \frac{\lambda_{0}}{2} [3\sigma^{2} - f_{0}^{2} + Q(M_{\pi}, T)]. \qquad (3.66)$$

3.8 Regularisation and renormalisation

The function Q(m,T), defined in Eq. (3.57), can be written as

$$Q(m,T) = Q_0(m) + Q_T(m,T),$$
(3.67)

where

$$Q_0(m) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{1}{2E_{\mathbf{p}}} \tag{3.68}$$

is the vacuum contribution and

$$Q_T(m,T) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{1}{E_{\mathbf{p}}} \, \frac{1}{e^{\beta E_{\mathbf{p}}} - 1} \tag{3.69}$$

depends on the temperature. Specifically, $Q_T(m,0) = 0$. The integral in $Q_T(m,T)$ is UV-finite and does not need to be regulated. This, however, is not true for the integral in $Q_0(m)$. We therefore introduce a finite UV momentum cutoff Λ , and obtain

$$Q_0(m) = \frac{1}{4\pi^2} \int_0^{\Lambda} \mathrm{d}p \; \frac{p^2}{\sqrt{p^2 + m^2}} = \frac{1}{8\pi^2} \left[\Lambda \sqrt{\Lambda^2 + m^2} - m^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m^2}}{m} \right]. \tag{3.70}$$

Assuming that $\Lambda \gg m$, we get

$$Q_0(m) = I_1 + m^2 I_2 + P_0(m)$$
(3.71)

where

$$I_1 \equiv \frac{\Lambda^2}{8\pi^2}, \tag{3.72}$$

$$I_2 \equiv \frac{1}{16\pi^2} \ln \frac{4\Lambda^2}{\mu^2}$$
 (3.73)

and we have defined the function $P_0(m)$ as

$$P_0(m) \equiv \frac{m^2}{16\pi^2} \ln \frac{m^2 e}{\mu^2}.$$
(3.74)

Here, we have also introduced the *renormalisation scale* μ , which should not be confused with the chemical potential μ_I . This notation is unfortunate, but it is the standard one.

Finally, we define

$$P(m,T) \equiv P_0(m) + Q_T(m,T), \qquad (3.75)$$

so that $P(m,0) = P_0(m)$ and

$$Q(m,T) = I_1 + m^2 I_2 + P(m,T).$$
(3.76)

Eq. (3.59) can then be written in the following form:

$$\rho^2 = (f_0^2 - I_1) - \frac{H^2}{\mu_I^4} + 2\mu_I^2 \left(\frac{1}{\lambda_0} + \frac{1}{2}I_2\right) - P(\mu_I, T).$$
(3.77)

We now define the *renormalised* parameters

$$f^2 = f_0^2 - I_1, (3.78)$$

$$\frac{1}{\lambda} = \frac{1}{\lambda} + \frac{1}{2}I_2, \qquad (3.79)$$

and obtain the following expression for the pion condensate:

$$\rho^2 = f^2 - \frac{H^2}{\mu_I^4} + \frac{2\mu_I^2}{\lambda} - P(\mu_I, T).$$
(3.80)

If we apply the same renormalisation to the expressions for the pion and sigma masses, Eqs. (3.65) and (3.66), we obtain

$$M_{\pi}^{2} = \frac{\lambda}{2} [\sigma^{2} - f^{2} + P(M_{\pi}, T)], \qquad (3.81)$$

$$M_{\sigma}^{2} = \frac{\lambda}{2} [3\sigma^{2} - f^{2} + P(M_{\pi}, T)]. \qquad (3.82)$$

We now wish to express H and the renormalised parameters λ and f in terms of the measured particle masses m_{π} and m_{σ} . These masses have been measured in vacuum, that is, when $T = \mu_I = \rho = 0$ and $\sigma = f_{\pi}/\sqrt{N}$. Then,

$$m_{\pi}^{2} = \frac{\lambda}{2} \left[\frac{f_{\pi}^{2}}{N} - f^{2} + P_{0}(m_{\pi}) \right], \qquad (3.83)$$

$$m_{\sigma}^{2} = \frac{\lambda}{2} \left[\frac{3f_{\pi}^{2}}{N} - f^{2} + P_{0}(m_{\pi}) \right].$$
(3.84)

By setting the renormalisation scale to $\mu^2 = m_{\pi}^2 e$ we find that $P_0(m_{\pi}) = 0$, and λ and f retain their tree-level values,

$$\lambda = \frac{N(m_{\sigma}^2 - m_{\pi}^2)}{f_{\pi}^2},$$
(3.85)

$$f^2 = \frac{m_{\sigma}^2 - 3m_{\pi}^2}{N(m_{\sigma}^2 - m_{\pi}^2)} f_{\pi}^2.$$
(3.86)

To calculate H, we look at the gap equations in vacuum. Eq. (3.54) is then automatically satisfied, while Eq. (3.56) can be written as

$$M^{2} = \frac{\lambda}{2} \left[\frac{f_{\pi}^{2}}{N} - f^{2} + P_{0}(M) \right].$$
(3.87)

Upon comparison with Eq. (3.83), this shows that in vacuum, M is in fact the pion mass. (In Ref. [43] it is stated that the pion mass is given by M also for finite temperatures, as long as one is in the unbroken phase. This, however, is no longer true at nonzero isospin chemical potential.) The remaining gap equation, Eq. (3.55), then gives

$$H = \frac{m_\pi^2 f_\pi}{\sqrt{N}},\tag{3.88}$$

which means that H is also given by its tree-level value.

3.9 Phase diagram

We use the measured values $f_{\pi} = 93$ MeV for the pion decay constant and $m_{\sigma} = 600$ MeV for the sigma mass [16]. At the physical point we use the measured pion mass $m_{\pi} = 139$ MeV, while we take $m_{\pi} = 0$ in the chiral limit. (Note that we always use the *physical* pion mass to set the renormalisation scale, so that $\mu = \sqrt{e} \cdot 139$ MeV even in the chiral limit.)

As shown in Fig. 3.2, the pion condensate, given by Eq. (3.80), decreases monotonically and continuously as the temperature increases. This means that we have a second order phase transition at the temperature where $\rho = 0$. This temperature will depend on μ_I , and we can thus make the phase diagram in Fig. 3.3. (Note that this diagram does not follow the convention with solid lines for first order transitions and dashed lines for second order transitions. Rather, both lines are second order transitions that separate the broken phase, $\rho \neq 0$, from the symmetric phase, $\rho = 0$.) This diagram can be compared with Fig. 6 in Ref. [56], and we see that the diagrams are essentially the same.

In the chiral limit, we see that pion condensation occurs at low temperatures for all values of μ_I . At the physical point, however, pions only condense when the chemical potential is larger than a certain critical value. When T = 0, this critical chemical potential can be found analytically, by plugging Eqs. (3.85), (3.86) and (3.88) into the expression (3.80) for the squared pion condensate. Not unexpectedly, we find that the critical chemical potential is $\mu_{Ic} = m_{\pi}$.

At large μ_I —or, equivalently, at high temperatures—we see that the phase transition lines in the chiral limit and at the physical point meet. This is because the constant H is independent of both temperature and chemical potential, and its effect is therefore suppressed by thermal effects. From a more physical point of view, this reflects the fact that the pion mass gets small compared to the thermal energies involved.



Figure 3.2: The pion condensate as a function of temperature at the physical point (solid line) and in the chiral limit (dashed line). The condensate has been normalised to its value at T = 0 (ρ_0), and we have used $\mu_I = 200$ MeV.



Figure 3.3: Phase diagram of the O(N) linear sigma model at the physical point (solid line) and in the chiral limit (dashed line).

Chapter 4

The two-flavour NJL model

In this chapter we investigate the phase diagram of the two-flavour Nambu–Jona-Lasinio model in the chiral limit at finite temperature, finite baryon number chemical potential and finite isospin chemical potential. We find that pion condensation occurs at temperatures lower than about 200 MeV.

4.1 Introduction

The Nambu–Jona-Lasinio (NJL) model was proposed by Nambu and Jona-Lasinio in two papers from 1961 [58, 59] as a model of interacting nucleons. This was before QCD, and even quarks, were known, so obviously, confinement was not an issue. On the other hand, there were already indications of (approximate) chiral symmetry. Since this would imply (almost) massless nucleons, the problem was to find a model that could explain the large nucleon mass without breaking chiral symmetry. They therefore introduced a Lagrangian for the nucleon field ψ with a chirally symmetric four-fermion point-like interaction,

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m_0)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau\psi)^2\right],\tag{4.1}$$

where m_0 is a small bare nucleon mass, $\tau = (\tau_1, \tau_2, \tau_3)$ is the vector of Pauli matrices acting in isospin space and G is a coupling constant, the value of which is determined by experiment. The self-energy induced by the interaction generates a large effective mass M, which stays large even in the chiral limit $m_0 \to 0$.

In the 1970s, after it had become evident that the nucleons are *not* the most elementary particles, and after the ensuing development of QCD, the NJL model was reinterpreted as a model for quark-quark interactions [60]. Since it does not take confinement into account, it has limited applicability. There are, however, situations like the present one where chiral symmetry, and not confinement, is the prominent feature of QCD.

The phase diagram of two-flavour QCD has been investigated using NJL-type models before [61, 62, 63]. Recently, Ebert and Klimenko have studied pion condensation in the NJL model with two flavours and separate chemical potentials for up and down quarks at zero temperature [64, 65]. In this chapter and the following we confirm their calculations, and extend them to finite temperature.

For clarity we will in the following consider an arbitrary number of colours, N_c , but we have set $N_c = 3$ in all numerical results. For simplicity, all numerical calculations are done in the chiral limit, $m_0 = 0$.

4.2 The NJL Lagrangian

Even after the reinterpretation of the NJL model as a quark model, it is common to use the Lagrangian shown in Eq. (4.1), with ψ being a quark field with two flavour and three colour degrees of freedom. This, however, is not the only possible choice. From a modern point of view,



Figure 4.1: In the NJL model the one-gluon exchange is replaced by a four-fermion interaction.

it is natural to write

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{det}}, \tag{4.2}$$

where the various terms are

$$\mathcal{L}_{\text{free}} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m_0)\psi, \qquad (4.3)$$

$$\mathcal{L}_{\text{sym}} = G_1 \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2 + (\bar{\psi}\tau\psi)^2 \right], \tag{4.4}$$

$$\mathcal{L}_{det} = G_2 \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau\psi)^2 - (\bar{\psi}i\gamma^5\psi)^2 - (\bar{\psi}\tau\psi)^2 \right].$$
(4.5)

 $\mathcal{L}_{\text{free}}$ is just the free Dirac Lagrangian, while \mathcal{L}_{sym} contains all possible chirally symmetric local 4-point interaction terms. In the chiral limit, both are symmetric under the groups [66]

$$\operatorname{SU}(N_c)_{\operatorname{colour}} \times \operatorname{SU}(N_f)_L \times \operatorname{SU}(N_f)_R \times \operatorname{U}(1)_V \times \operatorname{U}(1)_A.$$
 (4.6)

Remember from Sec. 1.4.4 that the $U(1)_A$ symmetry of the QCD Lagrangian is broken in the quantum theory. To model this behaviour, we introduce the term \mathcal{L}_{det} , which was first suggested by 't Hooft [67, 68] to describe instanton induced interactions between quarks in QCD. It can be written in determinant form as

$$\mathcal{L}_{det} = 2G_2 \left[\det \left\{ \bar{\psi}(1+\gamma^5)\psi \right\} + \det \left\{ \bar{\psi}(1-\gamma^5)\psi \right\} \right], \tag{4.7}$$

and breaks the $U(1)_A$ symmetry of $\mathcal{L}_{\text{free}}$ and \mathcal{L}_{sym} while preserving the others.

Since we are only considering two quark flavours, we may in the following choose $G_1 = G_2 = \frac{1}{2}G$ and thus limit ourselves to the original NJL Lagrangian, Eq. (4.1). In the three-flavour case we would have to use two independent coupling constants G_1 and G_2 , because \mathcal{L}_{det} then becomes a six-body interaction [69].

In the NJL model the gluon degrees of freedom do not appear, since gluon exchange is modeled by a local four-fermion interaction. This means that the colour symmetry is global, not local like in QCD. We also note that it is a vector symmetry. Had the Lagrangian been invariant under $SU(N_c)_L \times SU(N_c)_R$ transformations, chiral symmetry breaking would cause the appearance of unwanted coloured Goldstone bosons [70].

4.3 Finite density

In a closed system consisting of only up and down quarks, we have two conserved charges: baryon number and isospin. To a conserved charge we can always assign a chemical potential. We choose to work with the quark number instead of the baryon number, and therefore use the quark number chemical potential μ . It is easy to show that this is related to the baryon number chemical potential by $\mu_B = 3\mu$. Additionally, we define $\delta\mu$ to be a half of the isospin chemical potential μ_I . μ and $\delta\mu$ will then depend upon μ_u and μ_d and vice versa, and we will now derive the relations between these quantities.

The total number density of quarks is given by

$$n = n_u + n_d = -\frac{\partial\Omega}{\partial\mu_u} - \frac{\partial\Omega}{\partial\mu_d},\tag{4.8}$$

where we used Eq. (2.83) in the last equality. Using the same equation in conjunction with the chain rule, we also get

$$n = -\frac{\partial\Omega}{\partial\mu} = -\left(\frac{\partial\mu_u}{\partial\mu}\frac{\partial\Omega}{\partial\mu_u} + \frac{\partial\mu_d}{\partial\mu}\frac{\partial\Omega}{\partial\mu_d}\right).$$
(4.9)

Comparing Eqs. (4.8) and (4.9), we see that

$$\frac{\partial \mu_u}{\partial \mu} = 1, \quad \frac{\partial \mu_d}{\partial \mu} = 1.$$
 (4.10)

The isospin density is given by $n_I = \frac{1}{2}(n_u - n_d)$. By the same reasoning as above, we obtain

$$\frac{\partial \mu_u}{\partial \mu_I} = \frac{1}{2}, \quad \frac{\partial \mu_d}{\partial \mu_I} = -\frac{1}{2}.$$
(4.11)

We then see that

$$\mu_u = \mu + \frac{1}{2}\mu_I = \mu + \delta\mu$$
 (4.12)

$$\mu_d = \mu - \frac{1}{2}\mu_I = \mu - \delta\mu.$$
(4.13)

Conversely, we can solve for μ and $\delta\mu$, obtaining

$$\mu = \frac{1}{2}(\mu_u + \mu_d), \qquad (4.14)$$

$$\delta \mu = \frac{1}{2}(\mu_u - \mu_d).$$
(4.15)

Furthermore, by defining

$$\delta n = -\frac{\partial \Omega}{\partial (\delta \mu)},\tag{4.16}$$

it is easily seen that

$$n_u = \frac{1}{2}(n+\delta n), \qquad (4.17)$$

$$n_d = \frac{1}{2}(n-\delta n).$$
 (4.18)

Adding the finite-density term $\mu_i n_i$ to the Lagrangian, we obtain

$$\mathcal{L} = \bar{\psi} \left[i \gamma^{\mu} \partial_{\mu} - m_0 + \mu \gamma^0 + \delta \mu \gamma^0 \tau_3 \right] \psi + G \left[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \tau_i \psi)^2 \right].$$
(4.19)

This is confirmed by performing the matrix multiplication (in isospin space) in the third and fourth term in Eq. (4.19):

$$\bar{\psi}\left[\mu\gamma^0 + \delta\mu\gamma^0\tau_3\right]\psi = \mu_u(\bar{u}\gamma^0u) + \mu_d(\bar{d}\gamma^0d) = \mu_u n_u + \mu_d n_d.$$
(4.20)

From Eq. (4.19) we see that a nonzero quark number chemical potential μ breaks Lorentz invariance, whereas a nonzero value of $\delta\mu$ breaks both Lorentz invariance and isospin symmetry.

4.4 Mean-field approximation

We will obtain the mean-field approximation in two ways: First we will follow Ref. [60], and explicitly ignore the fluctuations in the interaction terms. Thereafter we will employ an equivalent but not so intuitive method, using auxiliary fields like we did in Sec. 3.5.

4.4.1 Intuitive method

To obtain the mean-field approximation, we assume that the system is in thermodynamic equilibrium, and rewrite the interaction term as

$$(\bar{\psi}\psi)^2 = (\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle)^2 + 2\bar{\psi}\psi\langle\bar{\psi}\psi\rangle - \langle\bar{\psi}\psi\rangle^2$$
(4.21)

where $\langle \bar{\psi}\psi \rangle$ is the expectation value of the operator $\bar{\psi}\psi$. The fluctuations around this value are then given by the first term, $(\bar{\psi}\psi - \langle \bar{\psi}\psi \rangle)$. In the mean-field approximation this term is neglected, and we obtain

$$(\bar{\psi}\psi)^2 \simeq 2\bar{\psi}\psi\langle\bar{\psi}\psi\rangle - \langle\bar{\psi}\psi\rangle^2. \tag{4.22}$$

We do the same for the pseudoscalar interaction term, and obtain

$$(\bar{\psi}i\gamma^5\tau_i\psi)^2 \simeq 2\bar{\psi}i\gamma^5\tau_i\psi\langle\bar{\psi}i\gamma^5\tau_i\psi\rangle - \langle\bar{\psi}i\gamma^5\tau_i\psi\rangle^2.$$
(4.23)

Note that

$$\langle \bar{\psi}\gamma^5\tau_1\psi\rangle = \langle \bar{u}\gamma^5d\rangle + \langle \bar{d}\gamma^5u\rangle, \qquad (4.24)$$

$$\langle \bar{\psi}\gamma^5\tau_2\psi\rangle = -i\langle \bar{u}\gamma^5d\rangle + i\langle \bar{d}\gamma^5u\rangle, \qquad (4.25)$$

$$\langle \bar{\psi}\gamma^5\tau_1\psi\rangle = \langle \bar{u}\gamma^5u\rangle - \langle \bar{d}\gamma^5d\rangle.$$
(4.26)

The expectation values $\langle \bar{\psi}\gamma^5 \tau_1 \psi \rangle$ and $\langle \bar{\psi}\gamma^5 \tau_2 \psi \rangle$ have the same quantum numbers as the charged pions, and we can use the remaining U(1)_V symmetry to rotate away any nonzero value of $\langle \bar{\psi}\gamma^5 \tau_2 \psi \rangle$. The expectation value $\langle \bar{\psi}\gamma^5 \tau_3 \psi \rangle$, on the other hand, has the quantum numbers of the neutral pion. We wish to study only charged pion condensation here, and we therefore take this quantity to be zero as well. Then,

$$\mathcal{L} = \bar{\psi} \left[i \gamma^{\mu} \partial_{\mu} - m_0 + \mu \gamma^0 + \delta \mu \gamma^0 \tau_3 \right] \psi + G \left[2 \langle \bar{\psi} \psi \rangle \bar{\psi} \psi + 2 \langle \bar{\psi} i \gamma^5 \tau_1 \psi \rangle \bar{\psi} i \gamma^5 \tau_1 \psi \right] - G \left[\langle \bar{\psi} \psi \rangle^2 + \langle \bar{\psi} i \gamma^5 \tau_1 \psi \rangle^2 \right].$$
(4.27)

For notational simplicity we define the parameters

$$M \equiv m_0 - 2G\langle \bar{\psi}\psi \rangle, \quad \rho \equiv -2Gi\langle \bar{\psi}\gamma^5\tau_1\psi \rangle, \tag{4.28}$$

and thus obtain the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - M + \mu\gamma^{0} + \delta\mu\gamma^{0}\tau_{3} - i\gamma^{5}\tau_{1}\rho)\psi - \frac{(M - m_{0})^{2} + \rho^{2}}{4G}.$$
(4.29)

We see that when $M \neq 0$, chiral symmetry is spontaneously broken. M is therefore called a *chiral* condensate, and we also note that it plays the role of an effective (constituent) quark mass. The charged pion condensate ρ , on the other hand, breaks parity and isospin symmetry as well as chiral symmetry. It is important to note that while ρ has the same interpretation here as in chapter 3, M does not.

The Lagrangian is now quadratic in the fermion fields, and it is a simple matter to obtain the effective action

$$S_{\text{eff}} = -\text{Tr} \ln \left[\gamma^0(i\omega_n) + \boldsymbol{\gamma} \cdot \mathbf{p} + M - \mu\gamma^0 - \delta\mu\gamma^0\tau_3 + i\gamma^5\tau_1\rho\right] + \beta V \frac{(M-m_0)^2 + \rho^2}{4G}$$
(4.30)

by performing the Gaussian integration like we did in Sec. 2.3.5. The trace in Eq. (4.30) has to be taken over Dirac, colour and flavour space as well as frequency-momentum space.

4.4.2 Auxiliary field method

Next, we will again follow the procedure we used in Sec. 3.5. We introduce the auxiliary fields σ and π_i , where i = 1, 2, 3, and add the following term to the Lagrangian:

$$\mathcal{L}_{\text{aux}} = -\frac{1}{4G}(\sigma^2 + \pi_i \pi_i). \tag{4.31}$$

We have already argued that doing this will not change the physics of the model. We also argued that the auxiliary fields can be shifted by a constant term, and by letting

$$\sigma \quad \to \quad \sigma - 2G\bar{\psi}\psi, \tag{4.32}$$

$$\pi_i \quad \to \quad \pi_i - 2G\bar{\psi}i\gamma^5\tau_i\psi, \tag{4.33}$$

the quartic interaction terms are eliminated from the Lagrangian:

$$\mathcal{L} \to \mathcal{L} + \mathcal{L}_{aux} = \bar{\psi} \left[i \gamma^{\mu} \partial_{\mu} - m_0 + \mu \gamma^0 + \delta \mu \gamma^0 \tau_3 - \sigma - i \gamma^5 \tau_i \pi_i \right] \psi - \frac{1}{4G} (\sigma^2 + \pi_i \pi_i)$$
(4.34)

The Lagrangian is now quadratic in the fermion fields, so these can be easily integrated out. The Euclidean action is then

$$S = -\operatorname{Tr} \ln(\gamma^{0}(i\omega_{n}) + \boldsymbol{\gamma} \cdot \mathbf{p} + m_{0} - \mu\gamma^{0} - \delta\mu\gamma^{0}\tau_{3} + \sigma + i\gamma^{5}\tau_{i}\pi_{i})$$

+
$$\frac{1}{4G} \int_{0}^{\beta} \mathrm{d}\tau \int \mathrm{d}^{3}x \; (\sigma^{2} + \pi_{i}\pi_{i})$$
(4.35)

To allow for a chiral condensate and a charged pion condensate we introduce nonzero expectation values for the fields σ and π_1 . Expanding around these values, we write [27]

$$\sigma \rightarrow -2G\langle \bar{\psi}\psi \rangle + \sigma = M - m_0 + \sigma,$$
(4.36)

$$\pi_1 \rightarrow -2G\langle \bar{\psi}i\gamma^5\tau_1\psi\rangle + \pi_1 = \rho + \pi_1, \qquad (4.37)$$

where σ and π_i are now quantum fluctuating fields, and M and ρ are defined as in Eq. (4.28). To obtain the mean-field approximation, we ignore the fluctuation terms. The integration is then trivial, and we again get the effective action (4.30).

4.5 Thermodynamic potential

It is now straightforward to calculate the thermodynamic potential using $\Omega = S_{\text{eff}}/\beta V$. We obtain

$$\Omega = \Omega_{\rm class} + \Omega_{\rm corr}, \tag{4.38}$$

where the classical potential is

$$\Omega_{\rm class} = \frac{(M - m_0)^2 + \rho^2}{4G} \tag{4.39}$$

and the thermal corrections are given by

$$\Omega_{\text{corr}} = -\frac{1}{\beta V} \text{Tr} \ln \left[\gamma^0(i\omega_n) + \boldsymbol{\gamma} \cdot \mathbf{p} + M - \mu \gamma^0 - \delta \mu \gamma^0 \tau_3 + i \gamma^5 \tau_1 \rho \right]$$

$$= -T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \ln \det A(p^0 = i\omega_n), \qquad (4.40)$$

where

$$A(p) \equiv \gamma^{\mu} p^{\mu} + M - \mu \gamma^0 - \delta \mu \gamma^0 \tau_3 + i \gamma^5 \tau_1 \rho.$$
(4.41)

The operator A acts in both Dirac, flavour and colour space, so the determinant has to be taken over all these spaces. As an ordinary two-dimensional matrix, A then has $4 \cdot 2 \cdot N_c = 8N_c$ rows and columns, and we write it in block matrix form as

$$A = \begin{pmatrix} \gamma^{\mu}p^{\mu} + M - (\mu + \delta\mu)\gamma^{0} & i\gamma^{5}\rho \\ i\gamma^{5}\rho & \gamma^{\mu}p^{\mu} + M - (\mu - \delta\mu)\gamma^{0} \end{pmatrix} \otimes \mathbf{1}_{c}.$$
 (4.42)

 $\mathbf{1}_c$ is the identity matrix in colour space. The determinant is then

$$\det A(p) = \left\{ \left[p^0 - (E_{\rho}^- - \mu) \right] \left[p^0 - (E_{\rho}^+ - \mu) \right] \left[p^0 + (E_{\rho}^+ + \mu) \right] \left[p^0 + (E_{\rho}^- + \mu) \right] \right\}^{2N_c}.$$
 (4.43)

Here, we have defined $E_{\rho}^{\pm} \equiv \sqrt{(E^{\pm})^2 + \rho^2}$, $E^{\pm} \equiv E \pm \delta \mu$ and $E \equiv \sqrt{\mathbf{p}^2 + M^2}$. The dispersion relations of the quasiparticles are determined by the poles of the propagator, namely the points where det A = 0:

$$\begin{aligned}
 E_u &= E_{\rho}^{-} - \mu, \\
 E_d &= E_{\rho}^{+} - \mu, \\
 E_{\bar{u}} &= E_{\rho}^{+} + \mu, \\
 E_{\bar{d}} &= E_{\rho}^{-} + \mu. \end{aligned}$$
 (4.44)

To avoid unnecessary clutter in our calculations, we now define $\varepsilon_{u,d} = E_{u,d}$ and $\varepsilon_{\bar{u},\bar{d}} = -E_{\bar{u},\bar{d}}$. The determinant is then simply

$$\det A = \prod_{a} (p^0 - \varepsilon_a)^{2N_c}, \qquad a = u, d, \bar{u}, \bar{d}.$$

$$(4.45)$$

Inserting this into Eq. (4.40), we obtain

$$\Omega_{\rm corr} = -2N_c \sum_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3} T \sum_n \ln(i\omega_n - \varepsilon_a) = -2N_c \sum_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \xi_a, \qquad (4.46)$$

where

$$\xi_a \equiv T \sum_n \ln(i\omega_n - \varepsilon_a). \tag{4.47}$$

Symmetrising this expression, we obtain

$$\xi_a = \frac{1}{2}T \sum_{n=-\infty}^{\infty} \ln(\omega_n^2 + \varepsilon_a^2), \qquad (4.48)$$

and by using Eq. (2.93) we find that

$$\xi_a = \frac{1}{2}\varepsilon_a + T\ln(1 + e^{-\beta\varepsilon_a}). \tag{4.49}$$

We now use the identity

$$T\ln(1+e^{-\beta\varepsilon_a}) = -\varepsilon_a + T\ln(1+e^{\beta\varepsilon_a}), \qquad (4.50)$$

which allows us to write ξ_a in a form more suited for the explicitly negative antiparticle terms:

$$\xi_a = -\frac{1}{2}\varepsilon_a + T\ln(1 + e^{\beta\varepsilon_a}). \tag{4.51}$$

Putting it all together, we arrive at a familiar expression, reminiscent of the thermodynamic potential (2.120) of a free Fermi gas,

$$\Omega_{\text{corr}} = -2N_c \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ E_{\rho}^- + T \ln[1 + e^{-\beta(E_{\rho}^- - \mu)}] + T \ln[1 + e^{-\beta(E_{\rho}^- + \mu)}] + E_{\rho}^+ + T \ln[1 + e^{-\beta(E_{\rho}^+ - \mu)}] + T \ln[1 + e^{-\beta(E_{\rho}^+ + \mu)}] \right\}, \quad (4.52)$$

and thus obtain the mean-field thermodynamic potential of the two-flavour NJL model:

$$\Omega = \frac{(M-m_0)^2 + \rho^2}{4G} - 2N_c \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \\ \times \Big\{ E_{\rho}^- + T \ln[1 + e^{-\beta(E_{\rho}^- - \mu)}] + T \ln[1 + e^{-\beta(E_{\rho}^- + \mu)}] \\ + E_{\rho}^+ + T \ln[1 + e^{-\beta(E_{\rho}^+ - \mu)}] + T \ln[1 + e^{-\beta(E_{\rho}^+ + \mu)}] \Big\}.$$
(4.53)

Some special cases

In the limit $T \to 0$, two of the thermal factors in Eq. (4.53) reduce to step functions while the other two vanish. Assuming that $\mu \ge 0$, we obtain the zero-temperature thermodynamic potential

$$\Omega_{T=0} = \frac{(M-m_0)^2 + \rho^2}{4G} - 2N_c \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \times \left\{ E_{\rho}^+ + E_{\rho}^- + (\mu - E_{\rho}^+)\theta(\mu - E_{\rho}^+) + (\mu - E_{\rho}^-)\theta(\mu - E_{\rho}^-) \right\}, \qquad (4.54)$$

which is the same expression that was found in Ref. [64]. If we also take $\mu \to 0$ and $\delta \mu \to 0$, we obtain the vacuum thermodynamic potential

$$\Omega_{\rm vac} = \frac{(M - m_0)^2 + \rho^2}{4G} - 2N_c \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(E_{\rho}^+ + E_{\rho}^- \right). \tag{4.55}$$

4.6 Phase diagrams

To make the phase diagrams, we will use the chiral condensate M and the charged pion condensate ρ as order parameters. If $M \neq 0$, chiral symmetry is broken, and if $\rho \neq 0$ we are in a phase with broken chiral symmetry and pion condensation. The equilibrium values of these parameters are obtained by solving the following set of gap equations:

$$\frac{\partial \Omega}{\partial M} = 0, \qquad (4.56)$$
$$\frac{\partial \Omega}{\partial \Omega} = 0. \qquad (4.57)$$

$$\frac{\partial \Omega}{\partial \rho} = 0. \tag{4.57}$$

Differentiating Ω with respect to M and ρ we obtain the equations

$$0 = \frac{M - m_0}{2G} - 2N_c M \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \frac{E^-}{E_{\rho} E} \left[1 - f(E_{\rho}^- - \mu) - f(E_{\rho}^- + \mu) \right] + \frac{E^+}{E_{\rho}^+ E} \left[1 - f(E_{\rho}^+ - \mu) - f(E_{\rho}^+ + \mu) \right] \right\}, \quad (4.58)$$

$$0 = \frac{\rho}{2G} - 2N_c \rho \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \frac{1}{E_{\rho}^-} \left[1 - f(E_{\rho}^- - \mu) - f(E_{\rho}^- + \mu) \right] + \frac{1}{E_{\rho}^+} \left[1 - f(E_{\rho}^+ - \mu) - f(E_{\rho}^+ + \mu) \right] \right\},$$
(4.59)

which have to be solved numerically. The function f is the Fermi-Dirac distribution function,

$$f(\varepsilon) = \frac{1}{e^{\beta\varepsilon} + 1}.$$
(4.60)

In the chiral limit, these equations typically have three different types of solutions:

- 1. M = 0 and $\rho = 0$,
- 2. $M \neq 0$ and $\rho = 0$,
- 3. M = 0 and $\rho \neq 0$,

each of which represents a different phase. When $m_0 \neq 0$ the chiral condensate never vanishes, and it is expected that there will be a phase where $M \neq 0$ and $\rho \neq 0$ simultaneously. When all the solutions have been found, one selects the one where the thermodynamic potential is lowest; this is the global minimum.

In addition to the parameters M and ρ we also calculate the baryon density,

$$n_B = -\frac{\partial\Omega}{\partial\mu_B} = -\frac{1}{3}\frac{\partial\Omega}{\partial\mu} = -\frac{1}{3}\frac{\partial\Omega}{\partial\mu} = \frac{2N_c}{3}\int \frac{d^3p}{(2\pi)^3} \left\{ f(E_{\rho}^- - \mu) - f(E_{\rho}^- + \mu) + f(E_{\rho}^+ - \mu) - f(E_{\rho}^+ + \mu) \right\}.$$
 (4.61)

Although n_B is not an order parameter, it is still interesting to include it in our phase diagrams. A vanishing baryon density effectively means that the number of particles is equal to the number of antiparticles.

The momentum integrals in the potential, gap equations and baryon density are ultraviolet divergent, and we regulate them by introducing a three-dimensional UV cutoff momentum Λ . The coupling constant G will then be dependent on Λ through the Dyson equation [60]

$$M = m_0 + 8N_c G \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{M}{\sqrt{p^2 + M^2}},\tag{4.62}$$

which is actually just the gap equation (4.58) in vacuum. An alternative to using a hard cutoff could be to include a form factor that falls off rapidly at large momenta, thus making the integrals convergent [71].



Figure 4.2: Phase diagram at T = 0.

To investigate the stability of the model, we will use two different sets of parameters G and Λ . If there are significant differences in the phase diagrams when different parameter sets are used, this could be a sign that the theory is in need of some refinement. We will follow Ref. [64], where the following two sets are suggested:

In the first one, which we will use most extensively, the ultraviolet cutoff is $\Lambda = 650$ MeV and the constituent quark mass is M = 300 MeV¹. In the chiral limit $m_0 = 0$, Eq. (4.62) then yields $G = 5.01 \ (\text{GeV})^{-2}$. This parameter set will henceforth be denoted P1. The second parameter set, which we will denote P2, is included for comparison. Here we use the cutoff $\Lambda = 600$ MeV and the constituent quark mass M = 400 MeV, giving the coupling $G = 6.82 \ (\text{GeV})^{-2}$.

When computers are used to perform floating-point² calculations, they are never exact—the precision of a calculation is limited by the hardware or the software. Particularly, and most importantly for our problem, a number that would be zero if calculations were exact may turn out small but finite. For this reason, values of order 10^{-5} or less have been considered to be zero.

4.6.1 Zero temperature

Fig. 4.2 shows the phase diagram of the two-flavour NJL model at zero temperature. Here we have used P1, and one immediately sees that the results of Ebert and Klimenko (Fig. 1 of Ref. [64]) have been reproduced.

The regions A and B are regions with pion condensation, that is, where $\rho > 0$. In both regions the chiral condensate M vanishes. The difference between the two regions is that in A we have a finite baryon density, whereas in B the baryon density vanishes. This means that the thermodynamic potential in A depends on the chemical potential μ , while it is independent of μ in B.

It is worth noting that along the line $\delta \mu = 0$, i.e. the *y*-axis, the potential no longer depends on the condensates M and ρ separately, but rather on the combination $M^2 + \rho^2$. The potential then has the familiar "mexican hat" shape, with infinitely many equivalent minima, from which we can choose any one we wish. However, parity is conserved in QCD at zero isospin chemical potential.

 $^{^1\,{\}rm The}$ masses of the lightest hadrons, the proton and the neutron, are about 900 MeV. $^2\,{\rm Non-integer}$



Figure 4.3: Chiral condensate as a function of quark chemical potential μ for $\delta \mu = 0$ MeV (solid line), and pion condensate as a function of μ for $\delta \mu = 150$ MeV (dashed line) and 300 MeV (dotted line) at T = 0.

To enforce this requirement we let $\rho = 0$ along this line [65], and consequently we get a finite chiral condensate here.

In the regions C and D both the pion condensate ρ and the chiral condensate M vanish, which means that chiral symmetry is restored. The difference between C and D again lies in the μ dependence of the potential. In region C there is a finite baryon density, indicating that Ω depends on the chemical potential, whereas in D the baryon density vanishes. It should be noted, however, that the parts of the diagram where $\delta \mu > \Lambda$ are not necessarily to be trusted, as the cutoff marks the highest energy scale for which the theory is valid.

As expected, nowhere in the diagram do both ρ and M have finite values simultaneously. This is a consequence of taking the chiral limit, and is in accordance with the results of Refs. [56, 62].

Fig. 4.3 shows the value of the pion condensate for two different finite values of the isospin chemical potential, together with the chiral condensate for $\delta \mu = 0$. From this figure it is clear that the phase transition between the broken phase (A, B) and the symmetric phase (C, D) is first order. The region where the solid line bends downwards corresponds to the region A in the phase diagram.

In Fig. 4.4 we see the phase diagram at zero temperature for P2. Note that although the general structure of the diagram is very similar, the region of pion condensation is larger. Also, the region A has disappeared, which is in accordance with Ref. [64] where it is claimed that for M = 400 MeV a cutoff $\Lambda > 720$ MeV is needed for this region to exist.

4.6.2 Finite temperature

Figs. 4.5 and 4.6 show the phase diagram at T = 50 MeV and T = 150 MeV, respectively. As the temperature increases, we see that the phase with pion condensation (regions A and B) shrinks. Our calculations indicate that at temperatures higher than about 200 MeV pion condensation does not occur at all. (For an example, see Fig. 4.10, where the phase diagram is shown in the $\delta\mu$ -T plane for $\mu = 0$.) Secondly, we also note that the regions with finite baryon density (A and C) grow as the temperature increases, and eventually the regions B and D are suppressed altogether.



Figure 4.4: Phase diagram at T = 0, parameter set 2.

The transition between the phases A and C gradually goes from first to second order, as indicated by a dashed line instead of a solid one. As before, along the line $\delta \mu = 0$ we choose a finite value for the chiral condensate instead of the pion condensate, to ensure that the latter vanishes and parity is conserved.

In Figs. 4.7 and 4.8 we again show the pion condensate for finite isospin chemical potential along with the chiral condensate for $\delta \mu = 0$, for T = 50 MeV and T = 150 MeV, respectively. From these figures it is clear that the phase transition is first order for low temperatures and second order for high temperatures.

Finally, for comparison with Figs. 3.2 and 3.3, we show the pion condensate as a function of temperature for $\delta \mu = \mu_I/2 = 100$ MeV (Fig. 4.9) as well as the phase diagram in the $\delta \mu$ -T plane (Fig. 4.10). Both are shown for $\mu = 0$. At small isospin chemical potentials, there is a slight difference in critical temperature between the two models. However, as the critical temperature in the present case is nearly constant, this difference gets larger as $\delta \mu$ increases. Most importantly, though, we see that the phase transition is second order in both models, so they describe the same qualitative behaviour.



Figure 4.5: Phase diagram at T = 50 MeV.



Figure 4.6: Phase diagram at T = 150 MeV.



Figure 4.7: Chiral condensate as a function of quark chemical potential μ for $\delta \mu = 0$ MeV (solid line), and pion condensate as a function of μ for $\delta \mu = 150$ MeV (dashed line) and 300 MeV (dotted line) at T = 50 MeV.



Figure 4.8: Chiral condensate as a function of quark chemical potential μ for $\delta \mu = 0$ MeV (solid line), and pion condensate as a function of μ for $\delta \mu = 150$ MeV (dashed line) and 300 MeV (dotted line) at T = 150 MeV.



Figure 4.9: The pion condensate as a function of temperature for $\delta \mu = 100$ MeV and $\mu = 0$. The condensate has been normalised to its value at T = 0.



Figure 4.10: The phase diagram for $\mu = 0$.

Chapter 5

Neutral matter

In chapter 4 we used the NJL model to describe quark matter at finite temperature, quark number chemical potential and isospin chemical potential. Through varying the latter we could freely adjust the relative abundances of up and down quarks. In the present chapter we will restrict ourselves a bit, and consider matter that is *electrically neutral*. We find that this leads to a relation between the chemical potentials. At low temperatures and densities, we find a region in the phase diagram where pions condense, but also an extended region with a nonzero chiral condensate.

5.1 Introduction

Our best chance of finding deconfined quark matter in Nature is probably in the cores of compact stars. The existence of a condensate of pions or kaons at the core of such a star will have observable effects, such as enhanced *neutrino cooling* [72, 73], which is the process in which a star sheds thermal energy by neutrino emission.

Matter in a neutron star must on average be electrically neutral, so a neutrality constraint must be imposed. We also have to take into account weak decay processes [60], like

$$d \leftrightarrow u + e + \bar{\nu}_e. \tag{5.1}$$

This means that we must introduce a chemical potential for the electrons. We are considering stable matter, so we assume that neutrinos have time to leave the system; their chemical potential therefore vanishes and we ignore them altogether.

Since we assume that neutrinos can freely leave the system, lepton number is not conserved. Also, because of the possibility of weak decay, the number of up and down quarks are no longer separately conserved. We are then left with only two conserved charges, namely baryon number and electric charge. Hence, we can express the chemical potentials of electrons, up quarks and down quarks in terms of the quark number chemical potential μ and the electric charge chemical potential μ_Q in the following manner:

$$\mu_u = \mu + \frac{2}{3}\mu_Q, \tag{5.2}$$

$$\mu_d = \mu - \frac{1}{3}\mu_Q, \tag{5.3}$$

$$\mu_e = -\mu_Q. \tag{5.4}$$

To enforce electric neutrality we require that

$$Q = -\frac{\partial\Omega}{\partial\mu_Q} = 0. \tag{5.5}$$

Because of this constraint, the chemical potentials μ and μ_Q will, for a given temperature, depend on each other.

Bulk matter in a neutron star must also be colour neutral, and if the system is in a colour superconducting phase, this constraint must sometimes be imposed explicitly [74, 75]. In the

present thesis we only work with baryon chemical potentials that are small enough that there is no colour superconducting phase, and so we do not need to introduce such a constraint. Nor is there any need to include chemical potentials for colour charge.

Ebert and Klimenko have described the phase structure of neutral quark matter at zero temperature in Ref. [65]. In the present chapter we confirm their calculations and again extend them to finite temperature.

5.2 Gap equations

For simplicity, we assume that we can describe the electrons by a non-interacting Fermi gas. We then add the following term to the Lagrangian:

$$\mathcal{L}_e = \bar{\psi}_e (i\gamma^\mu \partial_\mu - m_e + \mu_e \gamma^0) \psi_e, \qquad (5.6)$$

where ψ_e denotes the electron field and m_e is the electron mass. The thermodynamic potential of the electrons is then given by Eq. (2.120):

$$\Omega_e = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ E_{\mathbf{p}} + T \ln[1 + e^{-\beta(E_{\mathbf{p}} - \mu_Q)}] + T \ln[1 + e^{-\beta(E_{\mathbf{p}} + \mu_Q)}] \right\},\tag{5.7}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m_e^2}$. Since we are assuming that the quarks are massless, it is only natural to do the same for the electrons. The integrals in Eq. (5.7) can then be evaluated analytically (see appendix A), and one obtains

$$\Omega_e = -\frac{\mu_Q^4}{12\pi^2} - \frac{\mu_Q^2 T^2}{6} - \frac{7\pi^2}{180} T^4.$$
(5.8)

The thermodynamic potential for the quarks is obtained using the same method as in chapter 4. Alternatively (and more quickly), it can be obtained by inserting the above expressions for μ_u and μ_d into Eqs. (4.14) and (4.15) and thereby realising that we have to make the replacements

$$\mu \quad \to \quad \tilde{\mu} = \mu + \mu_Q / 6 \tag{5.9}$$

$$\delta\mu \rightarrow \delta\tilde{\mu} = \mu_Q/2$$
 (5.10)

in Eq. (4.53). Whichever method we choose, we get the result

$$\Omega_{q} = \frac{(M-m)^{2} + \rho^{2}}{4G} - 2N_{c} \int \frac{\mathrm{d}^{3}p}{(2\pi)^{3}} \times \left\{ E_{\rho}^{-} + T \ln[1 + e^{-\beta(E_{\rho}^{-} - \tilde{\mu})}] + T \ln[1 + e^{-\beta(E_{\rho}^{-} + \tilde{\mu})}] + E_{\rho}^{+} + T \ln[1 + e^{-\beta(E_{\rho}^{+} - \tilde{\mu})}] + T \ln[1 + e^{-\beta(E_{\rho}^{+} + \tilde{\mu})}] \right\},$$
(5.11)

where we now have $E^{\pm} = E \pm \delta \tilde{\mu}$. As before, $E_{\rho}^{\pm} = \sqrt{(E^{\pm})^2 + \rho^2}$ and $E = \sqrt{\mathbf{p}^2 + M^2}$. The total thermodynamic potential is the sum of the contributions from the electrons and from the quarks:

$$\Omega = \Omega_e + \Omega_q. \tag{5.12}$$

In thermal equilibrium, the gap equations

$$\frac{\partial\Omega}{\partial M} = 0, \tag{5.13}$$

$$\frac{\partial \Omega}{\partial \rho} = 0 \tag{5.14}$$

must be satisfied. Since Ω_e does not depend on M or ρ these are just Eqs. (4.58) and (4.59) again, with the substitutions (5.9) and (5.10). The neutrality constraint (5.5) can be written

$$\frac{\partial \Omega_e}{\partial \mu_Q} + \frac{\partial \Omega_q}{\partial \mu_Q} = 0, \tag{5.15}$$

where the derivatives evaluate to

$$\frac{\partial\Omega_e}{\partial\mu_Q} = -\frac{\mu_Q^3}{3\pi^2} - \frac{\mu_Q T^2}{3},$$

$$\frac{\partial\Omega_q}{\partial\mu_Q} = -N_c \int \frac{d^3p}{(2\pi)^3} \left\{ -\frac{E^-}{E_{\rho}^-} + \left(\frac{1}{3} + \frac{E^-}{E_{\rho}^-}\right) f(E_{\rho}^- - \tilde{\mu}) - \left(\frac{1}{3} - \frac{E^-}{E_{\rho}^-}\right) f(E_{\rho}^- + \tilde{\mu}) + \frac{E^+}{E_{\rho}^+} + \left(\frac{1}{3} - \frac{E^+}{E_{\rho}^+}\right) f(E_{\rho}^+ - \tilde{\mu}) - \left(\frac{1}{3} + \frac{E^+}{E_{\rho}^+}\right) f(E_{\rho}^+ + \tilde{\mu}) \right\}.$$
(5.16)

 $f(\varepsilon)$ is given by Eq. (4.60). The baryon density is

$$n_B = -\frac{\partial\Omega}{\partial\mu_B} = -\frac{1}{3}\frac{\partial\Omega}{\partial\mu} = -\frac{1}{3}\frac{\partial\Omega_q}{\partial\tilde{\mu}}$$
$$= \frac{2N_c}{3}\int \frac{\mathrm{d}^3p}{(2\pi)^3} \left\{ f(E_{\rho}^- - \tilde{\mu}) - f(E_{\rho}^- + \tilde{\mu}) + f(E_{\rho}^+ - \tilde{\mu}) - f(E_{\rho}^+ + \tilde{\mu}) \right\}.$$
(5.18)

In the zero-temperature limit, Eq. (5.15) reduces to that obtained by Ebert and Klimenko [65]:

$$0 = \frac{\mu_Q^3}{3\pi^2} + \frac{N_c}{3} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ \theta(\tilde{\mu} - E_{\rho}^-) + \theta(\tilde{\mu} - E_{\rho}^+) - \frac{E^-}{E_{\rho}^-} 3\theta(E_{\rho}^- - \tilde{\mu}) + \frac{E^+}{E_{\rho}^+} 3\theta(E_{\rho}^+ - \tilde{\mu}) \right\}$$
(5.19)

5.3 Phase diagram

We mentioned above that for a given temperature, the chemical potentials μ and μ_Q are mutually dependent. We choose to use the quark number chemical potential as a free parameter, and calculate μ_Q , M and ρ by simultaneously solving the gap equations (5.13) and (5.14) and the neutrality equation (5.5). In vacuum, Eq. (5.13) again reduces to the Dyson equation (4.62), which means that we can use the same parameter sets as in chapter 4. Most calculations are therefore done using the parameter set P1, with P2 included for comparison.

When $\mu_Q = 0$, the thermodynamic potential depends on the combination $M^2 + \rho^2$, and therefore has infinitely many global minima in M, ρ -space. From Eq. (5.10) we see that μ_Q plays the role of the isospin chemical potential in chapter 4, and by the reasoning given in Sec. 4.6.1 we choose $\rho = 0$ and $M \neq 0$ in this case.

5.3.1 Zero temperature

Fig. 5.1 shows the electric charge chemical potential while Fig. 5.2 shows the condensates M and ρ , all as functions of the quark chemical potential at zero temperature. When these are compared with Figs. 1 and 2 of Ref. [65] it is clear that we have reproduced the results of Ebert and Klimenko.

For small chemical potentials, $\mu < \mu_{1c} = 301$ MeV, the electric charge chemical potential μ_Q vanishes, so $\rho = 0$ and $M \neq 0$. This means that we are in the chirally noninvariant phase with no pion condensation. The baryon density is zero, so the system is already neutral and there is no need to neutralise it with pions and electrons.

On the other hand, when $\mu_{1c} < \mu < \mu_{2c} = 324$ MeV we are in the chirally noninvariant phase with pion condensation, that is, M = 0 and $\rho \neq 0$. The baryon density is nonzero, and the electric charge of the baryons is neutralised by both charged pions and electrons.

Finally, for large chemical potentials, $\mu > \mu_{2c}$ both condensates are zero, which means that chiral symmetry is restored. The baryon density is nonzero and the baryon electric charge is neutralised solely by the presence of electrons.

5.3.2 Finite temperature

The phase diagram for neutral matter at finite temperature is shown in Fig. 5.3. We see that as the temperature increases, the critical chemical potentials μ_{1c} and μ_{2c} gradually decrease to become zero at $T_{1c} = 55$ MeV and $T_{2c} = 185$ MeV, respectively. Both phase transitions are first order for all temperatures, as one can see from Fig. 5.4. The baryon density remains zero in phase A, while it increases with both temperature and chemical potential in phases B and C.



Figure 5.1: Electric charge chemical potential as a function of quark chemical potential at T = 0.



Figure 5.2: Pion (solid line) and chiral (dashed line) condensates as a function of quark chemical potential at T = 0.


Figure 5.3: Phase diagram of neutral quark matter.



Figure 5.4: Pion condensate as a function of quark chemical potential at T = 0 (solid line), 50 MeV (dashed line) and 150 MeV (dotted line).



Figure 5.5: Phase diagram of neutral quark matter, P2.

5.3.3 Parameter set 2

Fig. 5.5 shows the phase diagram for the second parameter set, P2. We see that the general structure is the same as for P1, but both the phases A and B have been enlarged.

Note that when T = 0, there is no phase with pion condensation, which is in accordance with the results of Ref. [65]. Ebert and Klimenko suggested there that at finite temperature a region where $\rho \neq 0$ may again appear, and as the figure shows, this is indeed the case.

Chapter 6

Summary and outlook

6.1 Summary

In this thesis we have studied deconfined quark matter at finite temperature and finite isospin chemical potential μ_I using the O(N) linear sigma model and the NJL model as effective models for QCD, with emphasis on the phenomenon of pion condensation.

The O(N) model was studied in both the chiral limit and at the physical point, and we found that pions condense only for low temperatures. In the chiral limit there is a pion condensate for any nonzero value of μ_I , whereas at the physical point the isospin chemical potential, as expected, has to be larger than the pion mass. The transition between the broken and the symmetric phase was found to be second order for all values of μ_I . At high temperatures the difference between the chiral limit and the physical point becomes insignificant.

The phase diagram of the 2-flavour NJL model was investigated only in the chiral limit, at both finite isospin chemical potential and finite baryon number chemical potential μ_B . It was found that pion condensation occurs at low temperatures and small values of μ_B , while chiral symmetry is restored at high temperatures and large μ_B . When μ_I is zero, there is no pion condensation, however in this case—and only in this case—there is a nonzero chiral condensate. There is no phase where both the chiral condensate and the pion condensate are nonzero simultaneously; this is a consequence of taking the chiral limit.

Using the NJL model we also applied a charge neutrality condition, to investigate the properties of such matter as one may find in the cores of compact stars. To this end we also considered an ideal electron gas as part of the system. We then found an extended region in the μ -T plane with a nonzero chiral condensate, as well as a phase with pion condensation. Nor in this case did we find a phase where both condensates are nonzero simultaneously.

The phase diagrams of the NJL model were calculated using two different sets of parameters, and although the general structure of the diagrams is the same, there are differences. This is an indication that the results are not entirely robust, and that the theory is in need of some refinement.

6.2 Outlook

In the case of the linear sigma model, the next step obviously is to include next-to-leading order terms in the 1/N expansion. This complicates the calculations, since one then has to take a momentum-dependent self-energy term into account. See for example Ref. [43], where the calculations have been done at zero chemical potential.

As for the NJL model, the natural extension would be to calculate the phase diagram using finite current quark masses (similar to what was done in Ref. [61]), so that chiral symmetry is never restored. In that case one expects to find that pion condensation occurs for isospin chemical potentials greater that the pion mass. It would also be interesting to discover the properties of the Goldstone bosons that appear because of chiral symmetry breaking. This was done at zero temperature in Ref. [65]. Finally, one could refine the approximation by including 1/N corrections, where N is the number of colours or flavours [76, 77].

Appendix A

Thermodynamic potential of an ideal gas of massless fermions

We found the thermodynamic potential of an ideal Fermi gas in Sec. 2.3.5; it was

$$\Omega = -2\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ E_{\mathbf{p}} + T \ln[1 + e^{-\beta(E_{\mathbf{p}} - \mu)}] + T \ln[1 + e^{-\beta(E_{\mathbf{p}} + \mu)}] \right\},\tag{A.1}$$

where $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2}$ and *m* is the mass of the fermions. If m = 0, this expression reduces to

$$\Omega = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left\{ p + T \ln[1 + e^{-\beta(p-\mu)}] + T \ln[1 + e^{-\beta(p+\mu)}] \right\},\tag{A.2}$$

where $p \equiv |\mathbf{p}|$. The first term is simply an infinite vacuum term which we can ignore, and we are left with

$$\Omega = -\frac{1}{\pi^2} (I_1 + I_2 + I_3), \tag{A.3}$$

where

$$I_{1} \equiv \int_{0}^{\mu} \mathrm{d}p \ p^{2} T \ln[1 + e^{-\beta(p-\mu)}], \qquad (A.4)$$

$$I_2 \equiv \int_{\mu}^{\infty} dp \ p^2 T \ln[1 + e^{-\beta(p-\mu)}], \qquad (A.5)$$

$$I_3 \equiv \int_0^\infty dp \, p^2 T \ln[1 + e^{-\beta(p+\mu)}].$$
 (A.6)

Using Eq. (4.50), we find that

$$I_1 = \int_0^\mu \mathrm{d}p \; p^2 \left\{ \mu - p + T \ln[1 + e^{\beta(p-\mu)}] \right\}. \tag{A.7}$$

We then apply the identity

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n},$$
(A.8)

which allows us to write

$$I_1 = \frac{\mu^4}{12} + T \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-n\beta\mu} \int_0^\mu \mathrm{d}p \; p^2 e^{n\beta p}. \tag{A.9}$$

This integral is easy to evaluate; we obtain

$$I_1 = \frac{\mu^4}{12} + T^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \left[(n\beta\mu)^2 - 2n\beta\mu + 2 - 2e^{-n\beta\mu} \right].$$
 (A.10)

Expanding the logarithms in ${\cal I}_2$ and ${\cal I}_3$ in the same way, we find that

$$I_2 = T^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \left[(n\beta\mu)^2 + 2n\beta\mu + 2 \right], \qquad (A.11)$$

$$I_3 = T^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \left[2e^{-n\beta\mu} \right].$$
 (A.12)

Then,

$$I_1 + I_2 + I_3 = \frac{\mu^4}{12} + T^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \left[2(n\beta\mu)^2 + 4 \right],$$
(A.13)

and by using the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12},$$
(A.14)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -\frac{7\pi^4}{720},$$
(A.15)

we finally obtain

$$\Omega = -\frac{\mu^4}{12\pi^2} - \frac{\mu^2 T^2}{6} - \frac{7\pi^2}{180}T^4.$$
(A.16)

Appendix B

O(N) model numerical calculations

The following is the Mathematica program that was used to calculate the phase diagram of the O(N) linear sigma model. Make note of the following functions:

- rhop and rhoc calculate the pion condensate squared (Eq. (3.59)), at the physical point and in the chiral limit, respectively.
- rhoplot makes the plot in Fig. 3.2.
- tcp and tcc calculate the critical temperature at the physical point and in the chiral limit, respectively.
- phaseplot makes the phase diagram, Fig. 3.3

lsm.m

```
(* Free parameters *)
n = 4
start = 150.
(* Experimental quantities (MeV) *)
mpi = 139.
msi = 600.
fpi = 93.
(* Renormalisation scale *)
scale = mpi Sqrt[n]
(* Dimensionless auxiliary function *)
Jint[x_y] := y^2 / Sqrt[x^2 + y^2] / (Exp[Sqrt[x^2 + y^2]] - 1)
J[x_] := NIntegrate[Jint[x,y], {y,0,Infinity}]
(* Sum-integral *)
qT[m_,t_] := t^2 J[m/t] / (2 Pi^2)
qT[m_,0.] := 0.
p0[m_] := m^2 Log[m^2 / scale^2] / (16 Pi^2)
p0[0.] := 0.
p[m_,t_] := p0[m] + qT[m,t]
(* Physical point *)
lp = n (msi^2 - mpi^2) / fpi^2
fp = fpi<sup>2</sup> (msi<sup>2</sup> - 3 mpi<sup>2</sup>) / (msi<sup>2</sup> - mpi<sup>2</sup>) / n
hp = mpi^2 fpi / Sqrt[n]
(* Chiral limit *)
```

Appendix C

NJL model numerical calculations

The programs that were used to make the phase diagrams of the NJL model were written in C++. They make use of a home-made library called $lk_phasediagram$, which consists of two files: A header file, $lk_phasediagram.h$, and an implementation file, $lk_phasediagram.cpp$. This library again makes use of the *GNU Scientific Library*¹ (GSL) to solve gap equations. The phase diagram library defines two classes and one function:

- The class system_base is a base class for classes that define a physical system, and must be subclassed. Among other things, such a class must provide pointers to the thermodynamic potential function and the gap equations.
- In the structure run_config one specifies which parameters should be used for the axes of a diagram, and the range of these axes. Additionally, it is used to tune the minimisation routine.
- The function make_diagram solves the gap equations for each point in the phase diagram. The output is sent to an output stream (a file, for instance), and is suitable for plotting with for example *gnuplot*. The first two columns in the output table are the diagram axis values. Thereafter follow equilibrium values for the order parameters, and the last column is the value of the thermodynamic potential.

In the file system.cpp the actual system is defined in a subclass of system_base. Integration is performed using integration routines from the GSL. This file is again included at the start of the file gridcalc.cpp, in which a configuration structure is set up and the make_diagram function is called.

The following files are the ones that were used to make the phase diagram in the non-neutral case at T = 50 MeV, with $\delta\mu$ and μ as the diagram axes (Fig. 4.5).

lk phasediagram.h

```
#ifndef LK_PHASEDIAGRAM_H
#define LK_PHASEDIAGRAM_H
#include <iostream>
#include <gsl/gsl_vector.h>
#include <gsl/gsl_multiroots.h>
namespace lk {
    const double VERY_LARGE = 1e20;
    // Class that provides runtime parameter info, such as axis limits, etc.
    struct run_config {
```

¹http://www.gnu.org/software/gsl/

```
double axis1_min, axis1_max, axis1_step;
    int axis1_parameter_index;
    double axis2_min, axis2_max, axis2_step;
    int axis2_parameter_index;
    double solver_epsabs, solver_epsrel;
    int solver_max_iterations;
};
// Nasty hack to provide gap equations for the system.
typedef int (* pointer_to_gap_eqns) (const gsl_vector* bound_params,
                                      void* free_params,
                                      gsl_vector* retvals);
\ensuremath{//} The base class for classes that define the physical system.
//\ Contains the potential and gap equations. Does nothing, must be
// subclassed.
class system_base {
   public:
        virtual int num_bound_params();
        virtual int num_free_params();
        virtual double param_max(int param_no);
        virtual double param_min(int param_no);
        virtual int num_startpoints();
        virtual void get_startpoint(int n, gsl_vector* v);
        virtual double potential(const gsl_vector* bound_params, void* free_params);
        virtual pointer_to_gap_eqns gap_eqns();
};
// This command makes the diagram
int make_diagram(std::ostream& out, system_base& sys, run_config& run, double* params,
    const gsl_multiroot_fsolver_type* solver_type = gsl_multiroot_fsolver_dnewton);
```

```
}
```

#endif // ifndef LK_PHASEDIAGRAM_H

$lk_phasediagram.cpp$

```
for (int i=0; i<n_start; i++) {</pre>
    startpoint[i] = gsl_vector_alloc(n_bound);
    sys.get_startpoint(i, startpoint[i]);
}
double* param_max = new double[n_bound];
double* param_min = new double[n_bound];
for (int i=0; i<n_bound; i++) {
    param_max[i] = sys.param_max(i);
    param_min[i] = sys.param_min(i);
}
// Check that axis indices are valid
if (run.axis1_parameter_index < 0 || run.axis1_parameter_index >= n_free) {
    std::cerr << "Axis 1 parameter index out of bounds ("</pre>
              << run.axis1_parameter_index << ")."
              << std::endl;
    return 1;
}
if (run.axis2_parameter_index < 0 || run.axis2_parameter_index >= n_free) {
    std::cerr << "Axis 2 parameter index out of bounds ("</pre>
              << run.axis2_parameter_index
              << ")."
              << std::endl;
    return 2;
}
// Initialise solver
gsl_multiroot_fsolver* solver = gsl_multiroot_fsolver_alloc(solver_type, n_bound);
// Declare vectors
gsl_vector* start;
gsl_vector* x;
gsl_vector* dx;
gsl_vector* x_min = gsl_vector_alloc(n_bound);
// Other variables
double pot_val, pot_min;
int point, solver_iter, solver_status;
int i;
// Gap equations
gsl_multiroot_function gap_eqns;
gap_eqns.f = sys.gap_eqns();
gap_eqns.n = n_bound;
gap_eqns.params = params;
// Iterate through diagram points
for (double ax1=run.axis1_min; ax1<=run.axis1_max; ax1+=run.axis1_step) {</pre>
    params[run.axis1_parameter_index] = ax1;
    for (double ax2=run.axis2_min; ax2<=run.axis2_max; ax2+=run.axis2_step) {</pre>
        params[run.axis2_parameter_index] = ax2;
        // Iterate through starting points
        pot_min = VERY_LARGE;
        for (point=0; point<n_start; point++) {</pre>
            start = startpoint[point];
            // Solve gap eqns
            gsl_multiroot_fsolver_set(solver, &gap_eqns, start);
```

```
solver_iter = 0;
        solver_status = GSL_CONTINUE;
        try {
            do {
                 gsl_multiroot_fsolver_iterate(solver);
                 x = gsl_multiroot_fsolver_root(solver);
                 dx = gsl_multiroot_fsolver_dx(solver);
                 solver_iter++;
                 solver_status = gsl_multiroot_test_delta(dx, x,
                                     run.solver_epsabs, run.solver_epsrel);
            } while (solver_status == GSL_CONTINUE
                         && solver_iter <= run.solver_max_iterations);</pre>
        } catch(...) {
            std::cerr << "Exception thrown at ("</pre>
                       << ax1
                       << " , "
                       << ax2
                       << "), startpoint "
                       << point
                       << "."
                       << std::endl;
        }
        // If no root found, proceed at next starting point
        if (solver_status == GSL_CONTINUE) continue;
        // Check if potential at root is smaller than before
        try {
            pot_val = sys.potential(x, params);
            if (pot_val < pot_min) {</pre>
                 pot_min = pot_val;
                 gsl_vector_memcpy(x_min, x);
            }
        } catch(...) {
            std::cerr << "Exception thrown at ("</pre>
                       << ax1
                       << " , "
                       << ax2
                       << ")."
                       << std::endl;
        }
    }
    if (pot_min > (VERY_LARGE - 1)) {
        std::cerr << "No minimum found at ("</pre>
                  << ax1
                  << " , "
                   << ax2
                   << ")."
                   << std::endl;
    } else {
        out << ax1 << " " << ax2 << " ";
        for (i=0; i<n_bound; i++)</pre>
            out << gsl_vector_get(x_min, i) << " ";</pre>
        out << pot_min << '\n';</pre>
    }
}
                // End axis2 iteration
out << '\n';
                // Add isolines for gnuplot
                // End axis1 iteration
```

}

```
// Free memory
    gsl_vector_free(x_min);
    gsl_multiroot_fsolver_free(solver);
    for (int i=0; i<n_start; i++)</pre>
        gsl_vector_free(startpoint[i]);
    delete [] startpoint;
    delete [] param_max;
    delete [] param_min;
}
                        // End of function
// Virtual functions from system_base
int system_base::num_bound_params() {}
int system_base::num_free_params() {}
double system_base::param_max(int param_no) {};
double system_base::param_min(int param_no) {};
int system_base::num_startpoints() {}
void system_base::get_startpoint(int n, gsl_vector* v) {}
double system_base::potential(const gsl_vector* bound_params, void* free_params) {}
pointer_to_gap_eqns system_base::gap_eqns() {}
```

system.cpp

}

```
#include <cmath>
#include <gsl/gsl_integration.h>
#include <gsl/gsl_vector.h>
using namespace std;
// Physical parameters
const double G = 5.01;
const double CUTOFF = 0.65;
// Integration parameters
const double INT_EPSABS = 0.01;
const double INT_EPSREL = 1e-4;
const size_t INT_POINTS = 1000;
// Free-parameter labels
const int DELTAMU = 0;
const int MU = 1;
const int T = 2;
const int NUM_FREE_PARAMS = 3;
// Other constants
const double THREE_PISQ = 3. / (M_PI*M_PI);
const double ONE_2G = .5 / G;
const double ONE_4G = .25 / G;
// Parameter structure for integrands
struct intparams {
    double m_sq;
    double delta_sq;
```

```
double deltamu;
    double mu;
    double temp;
                       // Set to 1/temp
    double beta;
};
// Three functions which we define further below.
double pot_int(double p, void* params);
int dpot(const gsl_vector* bound_params, void* free_params, gsl_vector* retvals);
double dpot_dm_int(double p, void* params);
double dpot_ddelta_int(double p, void* params);
// Class defining physical system
class finite_temp_system : public lk::system_base {
    public:
        int num_bound_params() { return 2; }
        int num_free_params() { return 3; }
        double param_max(int n) { return 1.0; }
        double param_min(int n) { return 0.0; }
        int num_startpoints() { return 4; }
        void get_startpoint(int n, gsl_vector* v) {
            switch(n) {
                case 0:
                    gsl_vector_set(v, 0, 0.);
                    gsl_vector_set(v, 1, 0.);
                    break;
                case 1:
                    gsl_vector_set(v, 0, 0.);
                    gsl_vector_set(v, 1, 0.5);
                    break;
                case 2:
                    gsl_vector_set(v, 0, 0.5);
                    gsl_vector_set(v, 1, 0.5);
                    break;
                case 3:
                    gsl_vector_set(v, 0, 0.5);
                    gsl_vector_set(v, 1, 0);
                    break;
            }
        }
        lk::pointer_to_gap_eqns gap_eqns() { return &dpot; }
        double potential(const gsl_vector* bound_params, void* free_params) {
            double* par = reinterpret_cast<double*>(free_params);
            double m = gsl_vector_get(bound_params, 0);
            double delta = gsl_vector_get(bound_params, 1);
            intparams ip;
            ip.m_sq = m*m;
            ip.delta_sq = delta*delta;
            ip.deltamu = par[DELTAMU];
            ip.mu = par[MU];
            ip.temp = par[T];
            ip.beta = 1/par[T];
            gsl_function f;
            f.function = &pot_int;
```

```
f.params = &ip;
            double integral, abserr;
            gsl_integration_workspace* iws = gsl_integration_workspace_alloc(INT_POINTS);
            gsl_integration_qag(&f, 0, CUTOFF, INT_EPSABS, INT_EPSREL, INT_POINTS,
                                GSL_INTEG_GAUSS41, iws, &integral, &abserr);
            gsl_integration_workspace_free(iws);
            return ONE_4G*(ip.m_sq + ip.delta_sq) - THREE_PISQ*integral;
       }
};
// Potential integrand
double pot_int(double p, void* params) {
    intparams par = *(intparams*) params;
    double e = sqrt(p*p + par.m_sq);
    double em = e - par.deltamu;
    double ep = e + par.deltamu;
    double emd = sqrt(em*em + par.delta_sq);
    double epd = sqrt(ep*ep + par.delta_sq);
    double lmm = par.temp * log(1 + exp(-par.beta*(emd-par.mu)));
    double lmp = par.temp * log(1 + exp(-par.beta*(emd+par.mu)));
    double lpm = par.temp * log(1 + exp(-par.beta*(epd-par.mu)));
    double lpp = par.temp * log(1 + exp(-par.beta*(epd+par.mu)));
    return p*p * (emd + lmm + lmp + epd + lpm + lpp);
}
// Potential gradient function
int dpot(const gsl_vector* bound_params, void* free_params, gsl_vector* retvals) {
    double* par = (double*) free_params;
    double m = gsl_vector_get(bound_params, 0);
    double delta = gsl_vector_get(bound_params, 1);
    intparams ip;
    ip.m_sq = m*m;
    ip.delta_sq = delta*delta;
    ip.deltamu = par[DELTAMU];
    ip.mu = par[MU];
    ip.temp = par[T];
    ip.beta = 1 / par[T];
    gsl_function f_dm, f_ddelta;
    f_dm.function = &dpot_dm_int;
    f_ddelta.function = &dpot_ddelta_int;
    f_dm.params = f_ddelta.params = &ip;
    double int_dm, int_ddelta, abserr;
    gsl_integration_workspace* iws = gsl_integration_workspace_alloc(INT_POINTS);
    gsl_integration_qag(&f_dm, 0, CUTOFF, INT_EPSABS, INT_EPSREL, INT_POINTS,
                        GSL_INTEG_GAUSS41, iws, &int_dm, &abserr);
    gsl_integration_qag(&f_ddelta, 0, CUTOFF, INT_EPSABS, INT_EPSREL, INT_POINTS,
                        GSL_INTEG_GAUSS41, iws, &int_ddelta, &abserr);
    gsl_integration_workspace_free(iws);
    gsl_vector_set(retvals, 0, m*(ONE_2G - THREE_PISQ * int_dm));
    gsl_vector_set(retvals, 1, delta*(ONE_2G - THREE_PISQ * int_ddelta));
```

```
return 0;
}
// Potential grad. integrand (1)
double dpot_dm_int(double p, void* params) {
    intparams par = *(intparams*) params;
    double e = sqrt(p*p + par.m_sq);
    double em = e - par.deltamu;
    double ep = e + par.deltamu;
    double emd = sqrt(em*em + par.delta_sq);
    double epd = sqrt(ep*ep + par.delta_sq);
    double fmm = 1 / (1 + exp(par.beta*(emd-par.mu)));
    double fmp = 1 / (1 + exp(par.beta*(emd+par.mu)));
    double fpm = 1 / (1 + exp(par.beta*(epd-par.mu)));
    double fpp = 1 / (1 + exp(par.beta*(epd+par.mu)));
    return p*p * ((em/(emd*e))*(1-fmm-fmp) + (ep/(epd*e))*(1-fpm-fpp));
}
// Potential grad. integrand (2)
double dpot_ddelta_int(double p, void* params) {
    intparams par = *(intparams*) params;
    double e = sqrt(p*p + par.m_sq);
    double em = e - par.deltamu;
    double ep = e + par.deltamu;
    double emd = sqrt(em*em + par.delta_sq);
    double epd = sqrt(ep*ep + par.delta_sq);
    double fmm = 1 / (1 + exp(par.beta*(emd-par.mu)));
    double fmp = 1 / (1 + exp(par.beta*(emd+par.mu)));
    double fpm = 1 / (1 + exp(par.beta*(epd-par.mu)));
    double fpp = 1 / (1 + exp(par.beta*(epd+par.mu)));
    return p*p * ((1/emd)*(1-fmm-fmp) + (1/epd)*(1-fpm-fpp));
}
```

gridcalc.cpp

```
#include <lk_phasediagram.h>
#include <fstream>
#include <iostream>
#include "system.cpp"
using namespace std;
// Alternative GSL error handler.
// Included to prevent program from quitting everytime a function
// can't be integrated.
void errhandler(const char* reason, const char* file, int line, int gsl_errno) {
    std::cerr << "GSL error " << gsl_errno << ": " << reason << std::endl;
}</pre>
```

```
int main() {
   // Use the above error handler
   gsl_set_error_handler(&errhandler);
   // Configure run
   lk::run_config rc;
   rc.axis1_parameter_index = DELTAMU;
   rc.axis1_min = 0.;
   rc.axis1_max = 1.;
   rc.axis1_step = 0.01;
   rc.axis2_parameter_index = MU;
   rc.axis2_min = 0.;
   rc.axis2_max = 0.35;
   rc.axis2_step = 0.0035;
   rc.solver_epsabs = 1e-4;
   rc.solver_epsrel = 1e-4;
   rc.solver_max_iterations = 10;
   // Set up free parameters. We only specify one here, since
   // we have three parameters and two are the diagram axes.
   double* fp = new double[NUM_FREE_PARAMS];
   fp[T] = 0.05;
                       // Set temperature to 50 MeV
   // Load system
   finite_temp_system sys;
   // Open file
   ofstream out("grid");
   if (!out) {
        cerr << "Error: Unable to open file 'grid' for write.";</pre>
       return 1;
   }
   // Make phase diagram
   lk::make_diagram(out, sys, rc, fp);
   // Free memory
   delete [] fp;
```

```
}
```

Bibliography

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