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Suggested solution for 2018 Exam in Electricity and Magnetism

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the detailed steps of the calculations.

Problem 1: Electric charge and electric field (8 points)
The charge density on the surface of the disk is $\sigma = Q/\pi R^2$. First, we need to know what the contribution to the electric field at the center is from a ring with charge Q' with radius r. Following the calculations in example 21.9 in YF, we find that

$$E_x = \int dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda x}{(x^2+r^2)^{3/2}}\int_0^{2\pi r} ds = \frac{Q'x}{4\pi\varepsilon_0(x^2+r^2)^{3/2}}. \tag{1}$$

where dE_x is the contribution from a segment ds on the ring and λ is the line charge density of the ring (so that $Q' = 2\pi r\lambda$). Knowing this, we can now compute the total electric field from the disk by dividing it into infinitesimal rings as shown in example 21.11 in YF. By replacing the ring charge Q' in Eq. (1) with the charge $dQ = 2\pi\sigma r dr$ of a ring with radius r on the uniformly charged disk and then integrating over the entire disk, we find:

$$E_x = \frac{\sigma x}{4\varepsilon_0}\int_0^R \frac{2rdr}{(x^2+r^2)^{3/2}}. \tag{2}$$

This integral can be looked up or performed by substituting $u = x^2 + r^2$, providing the final result

$$E_x = \frac{\sigma}{2\varepsilon_0}\Big[1 - \frac{1}{\sqrt{(R^2/x^2)+1}}\Big]. \tag{3}$$

Thus, the field points in the x-direction and has a magnitude given by Eq. (3) evaluated at $x = x_0$.

If we replaced the disk by a point particle Q, we would expect the field in that case to be larger because x_0 would be closer to the total charge than in the disk case: in the latter case, the charge Q is spread out on a larger area. Only in the limit $x \gg R$ would the disk and point particle start to produce equal electric fields. This can be seen by noting that for $x \gg R$, we have:

$$\frac{\sigma}{2\varepsilon_0}\Big[1 - \frac{1}{\sqrt{(R^2/x^2)+1}}\Big] = \frac{\sigma}{2\varepsilon_0}\big[1-(1-R^2/2x^2)+\ldots\big] = \frac{Q}{4\pi\varepsilon_0 x^2} - \ldots \tag{4}$$

showing that the correction to the $x \gg R$ limit for the disk will make its electric field *smaller* than for the point-particle.

Problem 2: Gauss' law (8 points)
Total charge given by

$$Q = \int_0^R 4\pi r^2\rho(r)dr. \tag{5}$$

Plug in $\rho(r)$ and obtain

$$Q = 4\pi\rho_0 R^3/12. \tag{6}$$

To obtain the electric field, we use Gauss' law. For radii larger than R, the field is the same as that of a point-particle with charge Q. For radii smaller than R, only part of the charge density is contained in the surface we integrate over. For $r < R$, we obtain

$$E \times 4\pi r^2 = Q_{\text{encl}}/\varepsilon_0 = \frac{1}{\varepsilon_0}\int_0^r 4\pi r'^2\rho_0(1-r'/R)dr'. \tag{7}$$

Performing the integral, one ends up with

$$E = \frac{Qr}{4\pi\varepsilon_0 R^3}\left(4 - \frac{3r}{R}\right) \tag{8}$$

for $r < R$.

Problem 3: Electric potential (8 points)
A conservative electric field means that the work done by the field only depends on the start and end point, not the particular path taken. Since $\mathbf{F} = q\mathbf{E}$, this means that we must be able to write the field as the gradient of a function V, in effect $\mathbf{E} = -\nabla V$. In this way, we see that $\int_A^B \mathbf{F}\cdot d\mathbf{r} = q[V(A) - V(B)]$ only depends on the start and end points A and B.

The relation between electric potential V and electric potential energy U is that $U = qV$ where q is the charge of the particle residing in the potential.

From the listed formulas, it follows that the potential created by a particle q_1 is $V = q_1/4\pi\varepsilon_0 r$ (when we choose zero potential at infinite distance). Therefore, the potential energy available when a particle q_2 resides at a distance r from q_1 is

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}. \tag{9}$$

We now have to apply this formula to all the pairs of particles in the cube. There are 12 pairs of q and $-q$ separated by a distance d. There are 12 pairs of equal charges separated by $\sqrt{2}d$. There are 4 pairs of q and $-q$ separated by $\sqrt{3}d$. In total, we thus obtain

$$U = \frac{q^2}{4\pi\varepsilon_0 d}\left(-12 + \frac{12}{\sqrt{2}} - \frac{4}{\sqrt{3}}\right). \tag{10}$$

Since the potential energy is less than zero, this configuration of charges is more beneficial energetically than the scenario where all charges are infinitely separated (which corresponds to zero potential energy). Thus, we can at least state that the charges should not repel each other and scatter to infinite separate distance.

Problem 4: Capacitance and dielectrica (8 points)
The students are supposed to know or be able to reason their way to the fact that when three capacitors are coupled in series, the charge on each plate must be the same and the effective capacitance C_{eff}, which satisfies $Q = C_{\text{eff}}V$, satisfies:

$$C_{\text{eff}} = \left(1/C_1 + 1/C_1 + 1/C_2\right)^{-1}. \tag{11}$$

The charge on condensator 2 is thus given by

$$Q = V\left(1/C_1 + 1/C_1 + 1/C_2\right)^{-1}. \tag{12}$$

The energy stored on a capacitor is given by $U = \frac{1}{2}QV$ and thus the total energy stored in the series coupling case is:

$$U_{\text{tot}} = \frac{1}{2}Q(V_1 + V_2 + V_3) = \frac{1}{2}QV \tag{13}$$

since the sum of all potential differences has to equal V and Q is given by Eq. (12).

In the parallell arrangement, the voltage difference across each capacitor must be the same. Thus, the charge on the two capacitors with C_1 has to be the same: Q_1. The potential difference across them is then

$$V_1 = Q_1/C_1. \tag{14}$$

The last capacitor C_2 has a different charge Q_2 and the potential difference is the same. Therefore,

$$Q_2 = Q_1\frac{C_2}{C_1}. \tag{15}$$

Problem 5: Current, resistance, and electromotive force (8 points)
The curves drawn by the student should at least vaguely resemble the ones in figure 25.6 in YF. For a metal, resistivity increases with temperature. For a semiconductor, it is reduced as temperature increases. For a superconductor, there is a sharp rise in resistivity at the critical temperature $T = T_c$ from zero. After that, it behaves like a metal.

One mechanism that causes resistance is scattering of electrons on phonons (lattice-vibrations). Another mechanism is scattering of electrons on defects in the material, i.e. deviations from a perfect crystal structure.

For a metal, ρ increases because more phonons become available to scatter on at higher temperatures T. For a semiconductor, the same thing happens, but this effect is countered by the fact that more conduction electrons become available as temperature increases (thermally excited from the valence band). This causes a net decrease in the resistivity.

The Fermi velocity is the instantaneous velocity of the electrons carrying the current and is typically of order $10^5 - 10^6$ m/s in a metal. The drift velocity is the net velocity with which the electrons carry current, i.e. taking into account the fact that they scatter. This velocity is much smaller, typically somewhere around $10^{-5} - 10^{-4}$ m/s. The students should be within 1 order of magnitude of these ranges in their responses.

Problem 6: DC circuits (8 points)
The power dissipated in the circuit is given by

$$P = \mathscr{E}^2/R_{\text{eff}} \tag{16}$$

where R_{eff} is the effective resistance of the circuit. Let us name R_{123} the effective resistance corresponding to the parallell coupling of the resistors R_1, R_2, R_3. We see that

$$1/R_{123} = 1/R_3 + 1/(R_1 + R_2). \tag{17}$$

Therefore, we have

$$R_{\text{eff}} = R_{123} + R_4. \tag{18}$$

Thus, we have to find the value of R_3 which causes

$$295 \text{ W } = (48 \text{ V})^2/(R_{123} + R_4). \tag{19}$$

From the above equation, it follows that we need

$$R_{123} + R_4 = 7.81\Omega. \tag{20}$$

Thus, $R_{123} = 4.81\Omega$. But from Eq. (17) we see that

$$R_3 = \frac{R_{123}(R_1 + R_2)}{R_1 + R_2 - R_{123}}. \tag{21}$$

Plugging in the given values for R_1 and R_2 as well as the obtained value for R_{123} gives us $R_3 = 12.1\Omega$.

Problem 7: Magnetic fields and magnetic forces (8 points)
There is no work performed by the magnetic field since it acts with a force on the particle that is perpendicular to its instantaneous motion.

A magnetic field has to have zero divergence since there exists no magnetic monopoles. Therefore, as listed in the formulas, we have $\nabla \cdot \mathbf{B} = 0$. This gives us the equation

$$\partial_y f(y) = 0. \tag{22}$$

Therefore, the most general form $f(y)$ can have is

$$f(y) = c_1 \tag{23}$$

where c_1 is a constant. **NB!** In the problem text in the exam, the field was said to be inhomogeneous. This could have caused confusion among students who subsequently might have discarded the constant solution assuming it would be wrong since the text specified "‘inhomogeneous”. Students who have reasoned in this manner in their response have been given a full score on this part of the problem.

Problem 8: Electromagnetic induction (8 points)
Otherwise, conservation of energy would have been violated. The resaon is that if the EMF created a flux which enhanced the external flux, then that flux increase would have spawned an even stronger EMF and so forth. In this way, the EMF would be ever-increasing, generating an infinite current.

A small strip of length W and width dr that is a distance r from the axis of the wire will have the following flux piercing it:

$$d\Phi = B(r)W dr = \frac{\mu_0 IW}{2\pi R^2} r dr. \tag{24}$$

The expression for $B(r)$ was obtained using Ampere's law for the uniformly distributed current. Thus, the total flux through the rectangle is obtained by integrating over all r:

$$\Phi = \int d\Phi = \frac{\mu_0 IW}{2\pi R^2}\int_0^R r dr = \frac{\mu_0 IW}{4\pi}. \tag{25}$$

Problem 9: Inductance (8 points)
Self-inductance is the phenomenon that a time-dependent current running through a closed circuit will induce a time-dependent flux through the circuit. This flux, in turn, causes an induced EMF according to Faraday's law. Thus, the net effect is that the current through the circuit has self-induced a new current. This new current flows in the opposite direction of the change in current that induced it in the first place (according to the argument of energy conservation that we discussed in problem 9).

When the switch is closed, the current has to be $I = 0^+$A through the inductors. It cannot have any finite value right after the switch is closed, because that would have corresponded to an infinite derivative di/dt which in turn would have caused an infinite induced EMF in the inductor. Instead, all current goes through the 15 Ω resistor. Therefore, the amperemeters show: $A_1 = A_4 = (25/55)$ A while $A_2 = A_3 = 0$ A.

After a long time, the circuit has reached its steady-state behavior and the current has stopped changing. Therefore, the voltage drop across each inductor is zero and they play no role. The three parallell-coupled resistors 5, 10, 15 Ω can then be combined into a single effective resistor $R_{\text{eff}} = (1/5 + 1/10 + 1/15)^{-1}\Omega = 2.73\Omega$. Therefore, the current through the A_1 amperemeter must be $I = 25/(40 + 2.73)$ A = 0.585 A since the effective resistance and the 40 Ω resistor are coupled in series.

Therefore, the voltage drop across each parallell branch in the circuit must be 25 V - 40 $\Omega \times 0.585$ A = 1.6 V since $40\Omega \times 0.585$ A is the voltage drop across the 40 Ω resistor. Since we now know the voltage drop across each branch, we can compute the resulting currents: A_2 will show a current (1.6/5) A = 0.32 A, A_3 will show a current (1.6/10) A = 0.16 A, while A_4 will show a current 1.6/15 A = 0.107 A.

Problem 10: AC circuits (8 points)
Inductive reactance X is defined as the ratio between the maximum amplitude V_0 of the time-dependent voltage across an inductor and the maximum amplitude I_0 of the resulting time-dependent current through the inductor:

$$X = V_0/I_0. \tag{26}$$

It can be thought of as a generalized resistance for time-dependent currents across an inductor. This "resistance" will depend on frequency. To see this, consider a current $I = I_0\cos(\omega t)$. Since $V = LdI/dt$, we obtain $V = -I_0\omega L\sin(\omega t) = I_0\omega L\cos(\omega t + \pi/2)$. We see that $V_0 = I_0\omega L$ and thus

$$X = \omega L. \tag{27}$$

Moreover, there is a phase-shift of $\pi/2$ between the voltage and the current: they are not in phase with each other. Since the reactance increases with ω, the inductor will (for a fixed voltage amplitude V_0) let a greater current through the smaller ω is.

Effectively, high frequencies are blocked while smaller frequencies are let through: a so-called low-pass filter.

Problem 11: Electromagnetic waves (8 points)
The key idea here is to compute the force that acts on you due to the intensity of the lightbeam shining in the opposite direction (carrying momentum away from the spaceship). We know that $F = pA$ where p is the pressure and A is the area the pressure acts upon. Now, the pressure exerted by light on an area is $p = p_{\text{radiation}} = I/c$ where I is the intensity of the lightbeam. But since the average effect carried by the lightbeam is $P_{\text{average}} = IA$, we see that $F = p_{\text{radiation}}A = IA/c = P_{\text{average}}/c$. Since we know that $P_{\text{average}} = 200$ W and $c = 3\times10^8$ m/s, we obtain that you will be accelerated according to Newton's 2. law:

$$a = F/m = P_{\text{average}}/(mc) = 4.44\times10^{-9}\text{m/s}^2. \tag{28}$$

Not a very large acceleration, but will it be enough to save you? The distance you will be able to cover with this acceleration (and zero initial velocity) is the standard expression:

$$x = \frac{1}{2}at^2 \tag{29}$$

(obtained by integrating $dx/dt = v(t) = at$). Inserting $x = 16$ m, we see that $t = \sqrt{2x/a} = 23.6$ hours. Thus, you *barely* make it back before running out of oxygen: only 0.4 hours = 24 minutes to spare.

Problem 12: Mixed topics (12 points)
(a) There is a change in the potential difference between the plates if we insert a dielectric medium because the dielectric medium becomes *polarized*. This induces a negative surface charge near the plate with the positive charge and vice versa. As a result, the *net* charge density near each plate is reduced and the electric field inside the dielectric is weakened. Thus, the potential difference is decreased.

(b) A paramagnetic material will orient its internal magnetic moments along the external magnetic field, whereas a diamagnetic material will create a magnetic moment in the opposite direction of the external field. Thus, the field is enhanced inside the material in the paramagnetic case and partially shielded in the diamagnetic case. The only thing diamagnetism and Faraday's law have in common is that the induced magnetic moment is opposite to the external one. Diamagnetism *cannot* be explained by Faraday's law, since diamagnetism occurs even for static magnetic fields.

(c) Examples are shown in figure 28.29 in the textbook by YF. Hysteresis is a type of memory effect which means that the magnetization of a material at a given external field B is not uniquely determined by the value of the field: it depends on the history of the material. For instance, if the material starts with a weak magnetization (due to the presence of domains) and one gradually increases the external field, the magnetization will continue to increase until it reaches saturation. If one then reverses the external field, the magnetization will start to diminish but will at $B = 0$ in general have a different value than it had originally. At some negative field $B < 0$, the magnetization will become zero and then reverse direction as the field continues to increase until it again reaches saturation in the negative direction.