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Suggested solution for 2020 Exam in Electricity and Magnetism

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the detailed steps of the calculations.

1: Power is given by $P = IV = I^2R$. The current is reduced by a factor 2 while the resistance over the light bulb is increased by a factor 2. Therefore, the power is reduced by a factor of 2, so that $P_2/P_1 = 0.5$.

2: When the capacitor is fully charged after a long time, no current will flow in the circuit. Moreover, there is no voltage drop over the inductor since $dI/dt = 0$ after a long time. Thus, the voltage drop over the capacitor has to equal the voltage supplied the EMS-source. Hence, $Q = V_0C$ and $I = 0$ after a long time.

3: The magnetic field at a distance y from the conductor is

$$B(y) = \mu_0 I/2\pi y. \tag{1}$$

It is directed out of the plane. The distance between the conductor and the square is increasing according to $d(t) = d_0 + vt$. Therefore, the induced flux in the square is time-dependent and equal to

$$\begin{aligned}\Phi(t) &= \int_d^{d+a} \frac{\mu_0 I}{2\pi y} a dy \\ &= \frac{\mu_0 I a}{2\pi} \ln\Big(\frac{d_0 + a + vt}{d_0 + vt}\Big).\end{aligned} \tag{2}$$

Differentiating with respect to time and taking absolute value, we find the induced EMS magnitude:

$$|\mathcal{E}| = \Big|\frac{\mu_0 I a}{2\pi}\Big[\frac{v}{d_0 + a + vt} - \frac{v}{d_0 + vt}\Big]\Big| \tag{3}$$

Using Lenz' law, we find the direction: the flux is reduced as the square moves, and hence a current is induced counterclockwise to attempt to counter this decrease in flux.

4: The current across the resistor is constant in time, $I_1 = V_0/R$. The capacitor is charged according to

$$Q(t) = V_0 C(1 - \mathrm{e}^{-t/2RC}) \tag{4}$$

which is found by using Kirchhoff's laws and solving the differential equation for charge. Therefore, the current in this part of the circuit is

$$I_2 = dQ/dt = \frac{V_0}{2R}(\mathrm{e}^{-t/2RC}). \tag{5}$$

The total current provided by the EMS-source is therefore $I = I_1 + I_2$ and the power delivered by the EMS-source is then time-dependent.

$$P(t) = V_0 I(t) = \frac{V_0^2}{R} + \frac{V_0^2}{2R}\mathrm{e}^{-t/2RC}. \tag{6}$$

The total energy delivered by the EMS-source between $t = 0$ and $t = t'$ is then:

$$U = \int_0^{t'} P(t)dt. \tag{7}$$

Putting in numbers, $U = 0.7059 \simeq 0.71$ J.

5: The plane is negatively charged, hence the E-field below the plane must point upward. The magnitude is found via Gauss' law, giving $E_0 = \sigma/2\varepsilon_0$.

The movement of the plane corresponds to a positive current along the x-axis. The current in the x-direction per unit length in the z-direction is $dI/dz = \sigma u_0$. Using Ampere's law gives $2B_0 l = \mu_0 \sigma u_0 l$ where l is the width along the z-direction of the Amperian loop that we use, and hence $B_0 = \mu_0 \sigma u_0/2$. The direction of the magnetic field is given by the right-hand rule, hence it is pointing in the negative z-direction.

6: The resonance condition for a parallell RLC circuit is the same as for a series RLC circuit and the impedance is the same in both cases: $Z = R$. However, the impedance is maximal at resonance in the parallell case whereas it is minimal in the series case. The false statement is that the current is out of phase with the voltage, because the circuit acts as if if the capacitor and inductor are not there when the frequency is at resonance, so that voltage and current are in phase across the resistor.

7: Magnetic flux through a closed surface is always zero, regardless of the current through the surface, since there are no magnetic monopoles.

8: The electric field between the plates is constant, so long as the distance between the plates is small compared to the area of the plates. Since the plates are considered to be infinitely large, the electric field does not change when d increases.

9: For a given medium/material, the speed of light has a certain value. This value is determined by $c = \lambda f$ where λ is the wavelength and f is the frequency. Thus, if one of them changes, the other one also has to change to keep c constant. Hence, they cannot vary independently of each other.

10: The magnetic field of a wire circulates the wire according to the right-hand rule: thumb in the direction of I will give fingers that bend in the direction of $\mathbf{B}$. Moreover, the contribution to the total field at a given point is strongest from the wire which is closest to that point. Using these two facts in the left region, the region between the wires, and the right region, one sees that alternative 3 on the drawing is correct.

11: The Fermi-velocity characterizes the motion of electrons which is present even without any current.

12: The reactance is given as the ratio of the maximum value of the voltage divided on the maximum value of the current, which the student should be able to find amounts to $\omega L/2$ by using $V = L' dI/dt$ with $V = V_0 \cos(\omega t)$ and $L' = L/2$.

13: Some key points which should be included in the reply in order to get a full score:

- Static charge distributions give rise to conservative electric fields
- Time-dep. magnetic fields give rise to non-conservative electric fields
- Difference between cons and non-cons fields is that the amount of work they do on a charge along a closed loop is zero and non-zero, respectively.

14: The reason can be found by examining what happens if the induced flux amplified the applied flux instead. Then, we would obtain an infinite amount of kinetic energy since the current would grow indefinitely. Hence, the only stable direction of current is the one corresponding to a voltage which counteracts the magnetic flux.

15: Even though an object is electrically neutral as a whole, it may have equally many positive as negative charges. When an electrically charged object is brought into vicinity of the neutral object, it will attract one type of charge in the neutral object and repel the other. This polarizes the neutral object, causing it to interact with the charged object.

16: The magnetic field from the wires is given by $B = \mu_0 I/2\pi r$ where r is the distance from the wire. When the currents go in the same direction, they can only cancel in the region between the wires, as seen when using the right-hand rule for the direction of the field. Let d_1 be the distance from the left wire to the point where they cancel and d_2 be the distance to that point from the right wire. We then have $d_1 + d_2 = 0.3$ m. Moreover, we have that

$$\mu_0 I_1/2\pi d_1 = \mu_0 I_2/2\pi d_2. \tag{8}$$

Combining these equations give $d_1 \simeq 13.3$ cm and $d_2 \simeq 16.7$ cm.

When the currents go in opposite direction, the region between the wires cannot have zero magnetic field since the individual fields from the wires add constructively there. To the left and right of the wires (along the x-axis), the field could potentially be zero. Since $I_1 > I_2$, we have to be closer to wire 2 in order to hope to achieve $B = 0$ since the individual fields scale with the current, $B_i \propto I_i$. This can only happen to the right of wire 2. Let x' be the distance to this point from wire 2. We then have

$$\frac{\mu_0 I_1}{2\pi(d+x')} = \frac{\mu_0 I_2}{2\pi x'}. \tag{9}$$

This is rearranged to

$$x' = I_2 d/(I_1 - I_2). \tag{10}$$

Putting in numbers give $x' = 1.2$ m, and hence $x = 1.35$ m.

17: The force between two charges in vacuum is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2} \tag{11}$$

when they are separated a distance d. We can rewrite this result as

$$F = \frac{q_1 q_2}{4\pi\varepsilon (d')^2} \tag{12}$$

where $d' = \sqrt{\varepsilon_0/\varepsilon} d$. Hence, we see that the same force acts between two particles separated by d in a medium ε_0 as the force which acts between two particles separated by $d' < d$ in a medium ε.

We can now use this knowledge to solve the problem, because we can now replace d_i in medium ε_i with $d_i\sqrt{\varepsilon_i/\varepsilon}$ and use ε_0 for the medium instead. The force is then

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 [d_1\sqrt{\varepsilon_1/\varepsilon_0} + d_2\sqrt{\varepsilon_2/\varepsilon_0}]^2}. \tag{13}$$

18: When $V(x \to \infty) = 0$, we can use that the contribution to the potential $V(x)$ from an infinitesimal charge dq on the wire, which corresponds to an infinitesimal segment dx', is:

$$dV = \frac{dq}{4\pi\varepsilon_0 (x - x')} \tag{14}$$

where

$$dq = \lambda dx' = \lambda_0 \sqrt{(x'+L)/(3L)} dx'. \tag{15}$$

To find the total potential in point $x > L$, we thus simply integrate over the entire wire:

$$V(x) = \int_{-L}^{L} dV = \frac{\lambda_0}{4\pi\varepsilon_0} \int_{-L}^{L} dx' \sqrt{\frac{x'+L}{3L}} \frac{1}{x-x'}. \tag{16}$$

You can look up the result in Rottmann or use some software to compute the integral analytically, with the result:

$$V(x) = \frac{\lambda_0}{4\pi\varepsilon_0}\sqrt{\frac{8}{3}}[\text{atanh}(\sqrt{2L/(x+L)} - 1]. \tag{17}$$

19: In a series coupling, the charge in each capacitor must be the same under equilibrium conditions (no current flowing). This can be understood by noting that under such conditions, no electric field can exist in the wires connecting V to the capacitors. Therefore, the first capacitor has to have a plate with charge Q and $-Q$, respectively. Now consider the plates of C_1

and C_2 which are connected to each other via a wire. Due to charge conservation, if the upper one has $-Q$, the lower one must have Q. Therefore, the lower plate of the second capacitor must have charge $-Q$ to prevent any electric field from existing in the wire.

The effective capacitance is defined as

$$C_{\text{eff}} = Q/V \tag{18}$$

and since we have:

$$V_1 = Q/C_1,\ V_2 = Q/C_2,\ V = V_1 + V_2, \tag{19}$$

it follows that

$$C_{\text{eff}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}. \tag{20}$$