1. **Wave Packet and Heisenberg Uncertainty Relations**

En electron is described via a wave-packet

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk \phi(k) e^{i[kx - \omega(k)t]}\,, \tag{1}$$

where the frequency is $\omega(k) = \hbar k^2/(2m)$. The expansion coefficients $\phi(k)$ have a large and narrow peak at $k = k_0$. The width of the peak is finite. We assume that the distribution of the wave vectors is Gaussian:

$$\phi(k) = \sqrt{\frac{\sigma}{2\pi\sqrt{\pi}}} \exp -\frac{\sigma^2}{2}(k - k_0)^2\,, \tag{2}$$

where σ is a finite and positive constant. What are the expectation values of the variance of the position $\Delta x = (\langle \hat{x}^2 \rangle - (\langle \hat{x} \rangle)^2)^{1/2}$ and the momentum $\langle \hat{p}_x \rangle$?

A: $\Delta x = 0$ and $\langle \hat{p}_x \rangle = \hbar k_0$.

B: $\Delta x = 0$ and $\langle \hat{p}_x \rangle = 0$.

C: $\Delta x = \sigma/\sqrt{2}$ and $\langle \hat{p}_x \rangle = \hbar k_0$.

D: $\Delta x = \sigma^2/\sqrt{2}$ and $\langle \hat{p}_x \rangle = \hbar k_0$.

E: $\Delta = 0$ and $\langle \hat{p}_x \rangle = \hbar k_0 - \hbar\sigma$.

Solution: C

From dimensional analysis of the distribution function of Eq. (2), we see that σ has dimensions of length since the dimension of the wave vector is inverse length. Therefore, options A, B, C, and E have the correct dimensions for the variance of the position. Furthermore, the dimension of the momentum must be equal to the dimensions of $\hbar k_0$. This rules out option E. Since the Gaussian is assumed to be narrow and is peaked around $k = k_0$ then the expectation of the momentum operator must be $\hbar k_0$. This rules out option B. Finally, it is stated this is a wave-packet with a finite distribution width, represented by the length σ. It is therefore unlikely that the variance of the position should vanish as in option A. The correct solution must therefore be option C.

Explicitly, we can compute the expectation value of the momentum:

$$\langle \hat{p}_x \rangle = \int_{-\infty}^{\infty} dx \Psi^*(x,t) \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi(x,t) \tag{3}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \hbar k \phi^*(k') \phi(k) \exp i(k-k')x \exp i(\omega(k) - \omega(k'))t \tag{4}$$

$$= 2\pi \int_{-\infty}^{\infty} dk \left| \phi(k) \right|^2 \hbar k \tag{5}$$

$$= \frac{\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} dk \hbar k \exp -\sigma^2 (k-k_0)^2 \tag{6}$$

Similarly, we can compute the expectation value of the position to the n-the power:

$$\langle \hat{x}^n \rangle = \int_{-\infty}^{\infty} dx \Psi^*(x,t) x^n \Psi(x,t) \tag{7}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' x^n \phi^*(k') \phi(k) \exp i(k-k')x \exp i(\omega(k) - \omega(k'))t \tag{8}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \phi^*(k') \phi(k) \left(-i \frac{\partial}{\partial k} \right)^n \exp i(k-k')x \tag{9}$$

$$= 2\pi \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \phi^*(k') \phi(k) \left(-i \frac{\partial}{\partial k} \right)^n \delta(k-k') \tag{10}$$

$$= 2\pi \int_{-\infty}^{\infty} dk \phi(k) \left(-i \frac{\partial}{\partial k} \right)^n \phi^*(k) \tag{11}$$

We then find that $\langle \hat{x} \rangle = 0$ and

$$\langle x^2 \rangle = \sigma^2/2 \tag{12}$$

so that $\Delta x = \sigma/\sqrt{2}$.

2. **Planck's Radiation Law**

Electromagnetic waves in a cavity satisfy the same boundary conditions as matter de Broglie waves. It can then be shown that the number of states dN in phase space $V d^3p$ is

$$dN = 2 \frac{1}{(2\pi\hbar)^3} V d^3p \,, \tag{13}$$

where $d^3p = 4\pi p^2 dp$, p is the momentum and V is the volume of the cavity. Photons are bosons and the probability to occupy a state at energy E at temperature T is

$$f_B(E,T) = \frac{1}{e^{E/(k_B T)} - 1} \,. \tag{14}$$

In this case, the energy density (energy per frequency and volume) of the electromagnetic waves can be expressed in terms of the frequency f, the temperature T, the speed of light c and Planck's constant h as

A:

$$u(f) = hf \frac{1}{e^{hf/(k_B T)} - 1}\,, \tag{15}$$

B:

$$u(f) = \frac{8\pi h}{c^3} \frac{f^3}{e^{hf/(k_B T)} - 1}\,, \tag{16}$$

C:

$$u(f) = \frac{c^2}{8\pi h} \frac{f}{e^{hf/(k_B T)} - 1}\,, \tag{17}$$

D:

$$u(f) = \frac{c}{8\pi h} \frac{f^5}{e^{hf/(k_B T)} - 1}\,, \tag{18}$$

E:

$$u(f) = \frac{8\pi h}{c^3} \frac{1}{e^{hf/(k_B T)} - 1}\,. \tag{19}$$

Solution: B

The relation between the energy and the momentum of a massless particle is $E = pc$, where c is the speed of light. The energy of a photon is $E = hf$, where h is Planck's constant and f is the frequency. We then find that $dp/df = hf/c$. The number of states in the frequency interval f to $f + df$ is then

$$dN = 2\frac{1}{(2\pi\hbar)^3} V 4\pi (\frac{h}{c})^3 f^2 df\,, \tag{20}$$

$$= \frac{8\pi}{c^3} V f^2 df\,. \tag{21}$$

Since the energy of one photon per volume is hf/V, we then find that the energy density of the photons (energy per frequency and volume) is

$$u(f) = \frac{hf}{V}\frac{dN}{df} f_B(hf, T)\,, \tag{22}$$

$$= \frac{8\pi h}{c^3} \frac{f^3}{e^{hf/(k_B T)} - 1}\,. \tag{23}$$

3. **Scattering by a Step Potential**

We consider a one-dimensional system, where there is no potential when $x < 0$. There is a finite and positive potential V_0 when $x > 0$. A particle is incident from the left with energy $E < V_0$. The wave function can be written as

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & ,\ x \le 0 \\ Ce^{-\kappa x} & ,\ x \ge 0 \end{cases}\,, \tag{24}$$

where $E = \hbar^2 k^2/(2m)$ and $V_0 - E = \hbar^2 \kappa^2/(2m)$. What is the correct form of the reflection amplitude r in this case?

A:

$$r = \frac{ik + \kappa}{ik - \kappa}\,, \tag{25}$$

B:

$$r = \frac{k - \kappa}{k + \kappa}\,, \tag{26}$$

C:

$$r = \frac{k + \kappa}{k - \kappa}\,, \tag{27}$$

D:

$$r = 1\,, \tag{28}$$

E:

$$r = \frac{ik}{\kappa + ik}\,. \tag{29}$$

Solution: A

The energy of the particle is less than the potential barrier it enters, $E < V_0$. It is therefore certain that the particle cannot continue as a propagating wave into the barrier. The reflection probability must therefore be equal to one, $R = |r|^2 = 1$. From their definitions, k and κ are real numbers. Therefore, only solutions A and D satisfy the condition that $R = 1$. For a barrier of a finite width, the particle can always penetrate slightly into the barrier causing a phase shift of the reflected wave. Hence, option D cannot be the correct solution and we are left with option A as the only possible choice.

We can also explicitly calculate the result. The potential is finite. In this case, the wave function and the derivative of the wave function must be continous everywhere, also at $x = 0$. At $x = 0$, we then find the continuity conditions

$$1 + r = C\,, \tag{30a}$$

$$1 - r = \frac{-\kappa C}{ik}\,. \tag{30b}$$

By subtracing (30b) from (30a), we can solve for the coefficient C and find

$$C = \frac{2ik}{ik - \kappa}\,. \tag{31}$$

In turn, by inserting the result for the coefficient C into (30a), we find

$$r = \frac{ik + \kappa}{ik - \kappa}\,. \tag{32}$$

4. **Planck's constant**

Planck's constant has the same units as

A: frequency,

B: the Hamiltonian,

C: angular momentum,

D: de Broglie wavelength,

E: momentum.

Solution: C

Recall that e.g. the eigenvalue of the orbital angular momentum is $L_z = m\hbar$, where m is a dimensionless quantum number.

5. **The Dipole Approximation**

The dipole approximation for the light-matter interaction uses the fact that

A all interactions on the atomic scale are suppressed by a factor α (the fine structure constant),

B: the charge distribution of all matter on the atomic scale can be very well approximated by its dipole moment,

C: the vector potential of all relevant field modes is approximately stationary on the atomic scale,

D: there is relativistic invariance,

E: we can use the Coulomb gauge.

Solution: B

In the dipole approximation, we utilize the fact that the wavelength of the electromagnetic field is typically much larger than the atomic size. It is then sufficient to approximate the charge distribution on the atomic scale by its dipole moment.

6. **The Lifetime of Excited States in the Hydrogen Atom**

The spontaneous transition rate associated with the light-matter interaction is

$$w_{if}^{(sp)} = \alpha \frac{4\omega_{if}^3}{3c^2} |\mathbf{d}_{fi}|^2 \,, \tag{33}$$

where $\alpha \approx 1/137$ is the fine structure constant. The dipole moment is

$$\mathbf{d}_{fi} = \int d\mathbf{r} \psi_f^*(\mathbf{r}) \mathbf{r} \psi_i(\mathbf{r}) \tag{34}$$

in terms of the initial state wave function $\psi_i(\mathbf{r})$ and the final state wave function $\psi_f(\mathbf{r})$.

We consider the lifetime of the low-excited states in the hydrogen atom. The energy difference between the levels is of the order of $10eV$. The Bohr radius is $a_0 = 0.5 \times 10^{-10}m$. The electron charge is $e \approx 1.6 \times 10^{-19}C$. Planck's constant is roughly $\hbar \sim 10^{-34} J \cdot s$. The speed of light is around $c \approx 3 \cdot 10^8 m/s$.

What is a rough estimate of the lifetime τ of the low-lying excited states?

A $\tau \sim 10^{-15}s$,

B: $\tau \sim 10^{-12}s$,

C: $\tau \sim 10^{-3}s$,

D: $\tau \sim 10^{-6}s$,

E: $\tau \sim 10^{-9}s$.

Solution: E

The dipole moment must be on the scale of the atom, $d \sim a_0$. The lifetime is proportional to the inverse spontaneous transition rate. We then find that

$$\tau \sim 1/w_{i\to f} \sim \left[\alpha \frac{\omega_{fi}^3}{c^2} |a_0|^2 \right]^{-1} \sim 10^{-9} s \,. \tag{35}$$

7. **The Position Representation in the General Formulation**

In the general formulation of quantum mechanics, we consider the position representation. In this case, $|x\rangle$ is an eigenvector to the position operator $\hat{x}$ with an eigenvalue x:

$$\hat{x}|x\rangle = x|x\rangle\,. \tag{36}$$

There is also a momentum operator $\hat{p}$. By using the canonical commutation relation, we can then find that

A

$$\langle x_2|\hat{p}|x_1\rangle = 0\,, \tag{37}$$

B:

$$\langle x_2|\hat{p}|x_1\rangle = \frac{\hbar}{i}\frac{\partial^2}{\partial x_2^2}\delta(x_2 - x_1)\,, \tag{38}$$

C:

$$\langle x_2|\hat{p}|x_1\rangle = \delta(x_2 - x_1)\,, \tag{39}$$

D:

$$\langle x_2|\hat{p}|x_1\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x_2}\delta(x_2 - x_1)\,, \tag{40}$$

E:

$$\langle x_2|\hat{p}|x_1\rangle = \frac{\hbar}{i}\delta(x_2 - x_1)\,. \tag{41}$$

In this problem it may or may not be useful to know the following property of the Dirac's delta-function (distribution), $x\delta'(x) = -\delta(x)$.

Solution: D

The canonical commutation relation is

$$[\hat{p}, \hat{x}] = \frac{h}{i}\,. \tag{42}$$

We can use the canonical commutation relation to find

$$\langle x_2|\,[\hat{p},\hat{x}]\,x_1\rangle = \langle x_2|\frac{\hbar}{i}|x_1\rangle\,, \tag{43}$$

$$\langle x_2|(\hat{p}\hat{x}-\hat{x}\hat{p}|x_1\rangle = \frac{\hbar}{i}\delta(x_2-x_1)\,, \tag{44}$$

$$\langle x_2|\hat{p}|x_1\rangle(x_1-x_2) = \frac{\hbar}{i}\delta(x_2-x_1)\,, \tag{45}$$

$$\langle x_2|\hat{p}|x_1\rangle = \frac{\hbar}{i}\frac{1}{(x_1-x_2)}\delta(x_2-x_1)\,, \tag{46}$$

$$\langle x_2|\hat{p}|x_1\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x_2}\delta(x_2-x_1)\,. \tag{47}$$

8. **The Momentum Representation in the General Formulation**

In the general formulation of quantum mechanics, we consider the momentum representation. In this case, $|p\rangle$ is an eigenvector to the momentum operator $\hat{p}$ with an eigenvalue p:

$$\hat{p}|p\rangle = p|p\rangle\,. \tag{48}$$

We then have that

A

$$\langle p_2|\hat{p}|p_1\rangle = 0\,, \tag{49}$$

B:

$$\langle p_2|\hat{p}|p_1\rangle = \frac{\hbar}{i}\frac{\partial^2}{\partial p_2^2}\delta(p_2-p_1)\,, \tag{50}$$

C:

$$\langle p_2|\hat{p}|p_1\rangle = \delta(p_2-p_1)\,, \tag{51}$$

D:

$$\langle p_2|\hat{p}|p_1\rangle = \frac{\hbar}{i}\frac{\partial}{\partial p_2}\delta(p_2-p_1)\,, \tag{52}$$

E:

$$\langle p_2|\hat{p}|p_1\rangle = p_2\delta(p_2-p_1)\,, \tag{53}$$

where $\hat{p}$ is the momentum operator. In this problem it may or may not be useful to know the following property of the Dirac's delta-function (distribution), $x\delta'(x) = -\delta(x)$.

Solution: E

We use that

$$\langle p_2|\hat{p}|p_1\rangle = \langle p_2|p_1|p_1\rangle\,, \tag{54}$$

$$= p_1\delta(p_1 - p_2)\,, \tag{55}$$

$$= p_2\delta(p_1 - p_2)\,. \tag{56}$$

9. **The Matrix Representation of Operators**

We assume that the state vector $|\psi\rangle$ can be expanded in an orthonormal basis $\{|k\rangle\}$:

$$|\psi\rangle = \sum_k \psi_k|k\rangle\,. \tag{57}$$

The state vector $|\psi\rangle$ can also be expanded in another orthonormal basis $\{|k'\rangle\}$:

$$|\psi\rangle = \sum_{k'} \psi_{k'}|k'\rangle\,. \tag{58}$$

We consider an operator $\hat{A}$ that can act on the state vector $|\psi\rangle$. We define the matrix elements $A_{ik} = \langle i|\hat{A}|k\rangle$ and $A'_{i'k'} = \langle i'|\hat{A}|k'\rangle$.

The relation between the matrix representations of the operator $\hat{A}$ in the two different representations is

A

$$A' = SAS\,, \tag{59}$$

B:

$$A' = SAS^{-1}\,, \tag{60}$$

C:

$$A' = S^{-1}AS^{-1}\,, \tag{61}$$

D:

$$A' = A\,, \tag{62}$$

E:

$$A' = SS^{-1}A\,, \tag{63}$$

where the elements of the matrix S are $S_{i'i} = \langle i'|i\rangle$ and $(S^{-1})_{ii'} = \langle i|i'\rangle$.

Solution: B

In the second basis, the matrix elements of the operator $\hat{A}$ are

$$A'_{i'k'} = \langle i'|\hat{A}|k'\rangle \tag{64}$$

We use the completeness relation

$$\sum_k |k\rangle\langle k| = 1 \tag{65}$$

twice in Eq. (64) to find

$$A'_{i'k'} = \sum_{ik} \langle i'|i\rangle\langle i|\hat{A}|k\rangle\langle k|k'\rangle\,, \tag{66}$$

$$A'_{i'k'} = \sum_{ik} S_{i'i}\langle i|\hat{A}|k\rangle(S^{-1})_{kk'}\,. \tag{67}$$

Since the matrix element of the operator $\hat{A}$ in the first basis is

$$A_{ik} = \langle i|\hat{A}|k\rangle \tag{68}$$

we find from Eq. (67) that the relations between the matrix representations of the operator $\hat{A}$ in the two different basis representations is

$$A' = SAS^{-1}\,. \tag{69}$$

10. **The Ladder Operators for a 1D Harmonic Oscillator**

We consider a 1D harmonic oscilator with the Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2\,, \tag{70}$$

where $\hat{p}$ is the momentum operator and $\hat{q}$ is the position operator. We introduce new operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\,, \tag{71}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\,. \tag{72}$$

The commutation relations between $\hat{a}$ and $\hat{a}^\dagger$ are then

A

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1\,, \tag{73}$$

B:

$$\left[\hat{a}^\dagger, \hat{a}\right] = 1\,, \tag{74}$$

C:

$$\left[\hat{a}, \hat{a}^\dagger\right] = \frac{\hbar}{i}\,, \tag{75}$$

D:

$$\left[\hat{a}, \hat{a}^\dagger\right] = \frac{i}{\hbar}\,, \tag{76}$$

E:

$$\left[\hat{a}, \hat{a}^\dagger\right] = 0\,. \tag{77}$$

Solution: A

The commutator between $\hat{a}$ and $\hat{a}^\dagger$ is

$$\left[\hat{a}, \hat{a}^\dagger\right] = \left[\sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}, \sqrt{\frac{m\omega}{s\hbar}}\hat{q} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\right]\,, \tag{78}$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = -2i\sqrt{\frac{m\omega}{2\hbar}}\frac{1}{\sqrt{2m\hbar\omega}}\left[\hat{q}, \hat{p}\right]\,, \tag{79}$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = -2i\frac{1}{2\hbar}\left(-\frac{\hbar}{i}\right)\,, \tag{80}$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1\,. \tag{81}$$

11. **The Number Representation for a 1D Harmonic Oscillator**

We consider a 1D harmonic oscilator with the Hamilton operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2\,, \tag{82}$$

where $\hat{p}$ is the momentum operator and $\hat{q}$ is the position operator. We introduce new operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\,, \tag{83}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\,. \tag{84}$$

In terms of these new operators, the Hamiltonian of Eq. (82) can be written as

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) . \tag{85}$$

In this case, $|n\rangle$ is an eigenstate for $\hat{H}$ with eigenvalue $E_n = \hbar\omega(n + 1/2)$:

$$\hat{H}|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle . \tag{86}$$

We assume that all the eigenstates are orthonormal. It can then be shown that

$$b_n|n-1\rangle = \hat{a}|n\rangle , \tag{87}$$

where b_n is a coefficient to be determined from the normalization conditions. Carrying out this task and choosing b_n to be real and positive, we find

A

$$b_n = \sqrt{n} , \tag{88}$$

B:

$$b_n = \sqrt{n + 1/2} , \tag{89}$$

C:

$$b_n = \sqrt{n + 1} , \tag{90}$$

D:

$$b_n = \sqrt{n + 3/2} , \tag{91}$$

E:

$$b_n = \sqrt{n + 2} . \tag{92}$$

Solution: A

The state $|n-1\rangle$ should be normalized:

$$\langle n-1|b_n^* b_n|n-1\rangle = |b_n|^2 , \tag{93}$$

$$\langle n|\hat{a}^\dagger \hat{a}|n\rangle = |b_n|^2 , \tag{94}$$

$$\langle n| \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2} \right) |n\rangle = |b_n|^2 , \tag{95}$$

$$\langle n| \left(n + \frac{1}{2} - \frac{1}{2} \right) |n\rangle = |b_n|^2 . \tag{96}$$

Since b_n is chosen to be real and positive, we find that

$$b_n = \sqrt{n}\,. \tag{97}$$

12. Angular Momentum

Quantum systems can have an angular momentum $\mathbf{J}$ that cannot be expressed in terms of the position $\mathbf{r}$ and the momentum $\mathbf{p}$ as the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

In general, angular momentum operators are defined in terms of the following commutation rules.

A

$$\left[\hat{J}_x, \hat{J}_y\right] = \frac{\hbar}{i}\hat{J}_z\,, \tag{98}$$

$$\left[\hat{J}_y, \hat{J}_z\right] = \frac{\hbar}{i}\hat{J}_x\,, \tag{99}$$

$$\left[\hat{J}_z, \hat{J}_x\right] = \frac{\hbar}{i}\hat{J}_y\,, \tag{100}$$

B:

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z\,, \tag{101}$$

$$\left[\hat{J}_y, \hat{J}_z\right] = i\hbar\hat{J}_x\,, \tag{102}$$

$$\left[\hat{J}_z, \hat{J}_x\right] = i\hbar\hat{J}_y\,, \tag{103}$$

C:

$$\left[\hat{J}_x, \hat{J}_y\right] = 0\,, \tag{104}$$

$$\left[\hat{J}_y, \hat{J}_z\right] = 0\,, \tag{105}$$

$$\left[\hat{J}_z, \hat{J}_x\right] = 0\,, \tag{106}$$

D:

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_x\,, \tag{107}$$

$$\left[\hat{J}_y, \hat{J}_z\right] = i\hbar\hat{J}_y\,, \tag{108}$$

$$\left[\hat{J}_z, \hat{J}_x\right] = i\hbar\hat{J}_z\,, \tag{109}$$

E:

$$\left[\hat{J}_x, \hat{J}_y\right] = \frac{\hbar}{i}\hat{J}_x\,, \tag{110}$$

$$\left[\hat{J}_y, \hat{J}_z\right] = \frac{\hbar}{i}\hat{J}_y\,, \tag{111}$$

$$\left[\hat{J}_z, \hat{J}_x\right] = \frac{\hbar}{i}\hat{J}_z\,. \tag{112}$$

Solution: B

The general angular momentum operators must satisfy the same commutation relations as the orbital angular momentum operator. In the position representation, we have

$$\hat{L} = \mathbf{r} \times \hat{\mathbf{p}} = \mathbf{r} \times \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}\,. \tag{113}$$

For the various components, we then get

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y\,, \tag{114}$$

$$\hat{L}_y = z\hat{p}_x - x\hat{p}_z\,, \tag{115}$$

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x\,. \tag{116}$$

The commutation relations between $\hat{L}_x$ and $\hat{L}_y$ are then

$$\left[\hat{L}_x, \hat{L}_y\right] = [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z]\ , \tag{117}$$

$$= [y\hat{p}_z, z\hat{p}_x] + [y\hat{p}_z, -x\hat{p}_z] \tag{118}$$

$$+ [-z\hat{p}_y, z\hat{p}_x] + [-z\hat{p}_y, -x\hat{p}_z] \tag{119}$$

$$= [y\hat{p}_z, z\hat{p}_x] + [-z\hat{p}_y, -x\hat{p}_z] \tag{120}$$

$$= \frac{\hbar}{i}\left(y\hat{p}_x - x\hat{p}_y\right)\ , \tag{121}$$

$$= i\hbar\hat{L}_z\,. \tag{122}$$

We can find the two other commutation relations by cyclic permutations:

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x\,, \tag{123}$$

$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y\,. \tag{124}$$

13. **The Electron Spin**

The electron spin can be represented in terms of the two eigenvectors corresponding to the angular momentum operators, $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$. The matrix representation of the basis vectors can then be represented by

$$|\uparrow\rangle = \begin{pmatrix} \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle \\ \langle \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle \end{pmatrix} \tag{125}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{126}$$

and

$$|\downarrow\rangle = \begin{pmatrix} \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle \\ \langle \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle \end{pmatrix} \tag{127}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{128}$$

In the matrix representation, the electron spin operator can be expressed in terms of the Pauli matrices $\sigma = (\sigma_x, \sigma_y, \text{and } \sigma_z)$:

$$\mathbf{S} = \frac{\boldsymbol{\hbar}}{\mathbf{2}} \boldsymbol{\sigma} \tag{129}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{130}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{131}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{132}$$

We assume that the spin state is an eigenstate of the electron spin operator along the x-direction. The normalized 2-component eigenstate ψ_x is then

A

$$\psi_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{133}$$

B:

$$\psi_x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{134}$$

C:

$$\psi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} , \tag{135}$$

D:

$$\psi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \tag{136}$$

E:

$$\psi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} , \tag{137}$$

Solution: D

The system is in an eigenstate ψ_x corresponding to the electron spin along the x-direction. Then, we must have

$$\frac{\hbar}{2}\sigma_x\psi_x = \hbar m_s \psi_x \,, \tag{138}$$

where m_s is the spin quantum number associated with the projection along the x-direction. To find the eigenstate

$$\psi_x = \begin{pmatrix} \psi_{x\uparrow} \\ \psi_{x\downarrow} \end{pmatrix} , \tag{139}$$

we must then find the eigenvalues of the equation

$$\hbar \begin{pmatrix} -m_s & \frac{1}{2} \\ \frac{1}{2} & -m_s \end{pmatrix} \begin{pmatrix} \psi_{x\uparrow} \\ \psi_{x\downarrow} \end{pmatrix} = 0 \,. \tag{140}$$

By requiring that the determinant of the matrix vanishes, we determine the eigenvalues:

$$m_s^2 - \left(\frac{1}{2}\right)^2 = 0 \tag{141}$$

so that $m_S = \pm 1/2$. Since the electron is in a state that corresponds to an eigenstate of the spin along the x-direction, we choose $m_s = 1/2$ and find for the components of the state that

$$-\psi_{x\uparrow} + \psi_{x\downarrow} = 0 \,. \tag{142}$$

A normalized solution is then

$$\psi_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{143}$$

14. **The Electron Spin in a Magnetic Field**

We consider the electron spin in an external magnetic field along the x-direction. In the matrix representation, the Hamiltonian can be represented by

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{144}$$

where ω is the Larmor frequency.

We assume that the initial state (at $t = 0$) is an eigenstate of the electron spin operator along the z-direction,

$$\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{145}$$

At later times ($t > 0$), the state will then evolve into

A:

$$\psi(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{146}$$

B:

$$\psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{pmatrix} , \tag{147}$$

C:

$$\psi(t) = \begin{pmatrix} 0 \\ -i \sin \omega t \end{pmatrix} , \tag{148}$$

D:

$$\psi(t) = \begin{pmatrix} \cos \omega t \\ 0 \end{pmatrix} , \tag{149}$$

E:

$$\psi(t) = \begin{pmatrix} \cos \omega t \\ -i \sin \omega t \end{pmatrix} . \tag{150}$$

Solution: E

The evolution of the eigenstates is determined by the Schrödinger equation:

$$H\psi(t) = i\hbar\frac{\partial}{\partial t}\psi(t) \tag{151}$$

We express the 2-component state as

$$\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} . \tag{152}$$

The time-dependent equation we must solve is then

$$\hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i\hbar\frac{\partial}{\partial t} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \tag{153}$$

In other words, we need to solve the time-dependent coupled equations

$$\hbar\omega b(t) = i\hbar\frac{\partial}{\partial t}a(t)\,, \tag{154a}$$

$$\hbar\omega a(t) = i\hbar\frac{\partial}{\partial t}b(t)\,. \tag{154b}$$

A general solution of Eq. (154) is

$$a(t) = A_+ e^{i\omega t} + A_- e^{-i\omega t}\,, \tag{155a}$$

$$b(t) = -A_+ e^{i\omega t} + A_- e^{-i\omega t}\,. \tag{155b}$$

At $t = 0$, we must have $a(t = 0) = 1$ and $b(t = 0) = 0$, which gives the following equations for the coefficients A_+ and A_-:

$$A_+ + A_- = 1\,, \tag{156a}$$

$$-A_+ + A_- = 0\,, \tag{156b}$$

with the solutions $A_+ = 1/2$ and $A_- = -1/2$ so that the state is

$$\psi(t) = \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix} . \tag{157}$$

15. **Addition of Angular Momenta of Two Spin** $1/2$ **Particles**

We consider the total spin angular momentum $\hat{\mathbf{S}}$ of two spin 1/2 particles, $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$. $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ are the spin operators associated with particle 1 and 2, respectively. The eigenvalue of $\hat{\mathbf{S}}^2$ is $\hbar^2 s(s+1)$ and the eigenvalue of $\hat{S}_z$ is $\hbar m$. The possible values of s and m are

A: $s = 0$ and $m = 0, \pm 1$ or $s = 1$ and $m = 0$,

B: $s = 0$ and $m = 0$ or $s = 1$ and $m = 0, \pm 1$,

C: $s = 1/2$ and $m = \pm 1/2$ or $s = -1/2$ and $m = \pm 1/2$,

D: $s = 0$ and $m = 0$ or $s = 1$ and $m = 1$,

E: $s = 0$ and $m = 0$ or $s = 1/2$ and $m = \pm 1/2$.

Solution: B

There are two spin 1/2 particles. The total spin quantum number s is then $s = 1/2 - 1/2 = 0$ or $s = 1/2 + 1/2 = 1$. When $s = 0$, we must have $m = 0$. When $s = 1$, we can have $s = 0, \pm 1$. There are two possibles states for each particle, so in total for the two spin 1/2 particle there are four possible states.

16. **Time-dependent Perturbation Theory**

We consider a system that is subject to a weak time-dependent perturbation described by the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \lambda \hat{V}(t) \,, \tag{158}$$

where $\hat{H}_0$ is time-independent and $\lambda \hat{V}(t)$ is the perturbation. We assume that we know the eigenstates of $\hat{H}_0$,

$$\hat{H}_0 |\psi_n^{(0)}(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi_n^{(0)}(t)\rangle \tag{159}$$

so that

$$|\psi_n^{(0)}(t)\rangle = e^{-iE_n t/\hbar} |\psi_n\rangle \,, \tag{160}$$

where $\{|\psi_n\rangle\}$ are stationary states with associated eigenenergies $\{E_n\}$:

$$\hat{H}_0 |\psi_n\rangle = E_n |\psi_n\rangle \,. \tag{161}$$

We expand the exact state $|\psi\rangle$ in terms of $|\psi_n^{(0)}(t)\rangle$ that form complete set

$$|\psi(t)\rangle = \sum_k a_k(t) e^{-iE_k t/\hbar} |\psi_k\rangle \,, \tag{162}$$

where $\{a_k(t)\}$ are time-dependent expansion coefficients. We can insert the expansion of Eq. (162) in the Schrödinger equation and then find the time-dependent equation for the expansion coefficients $a_k(t)$:

A:

$$i\hbar\frac{d}{dt}|a_n(t)|^2 = \sum_k |a_k(t)|^2 e^{i\omega_{nk}t}|\lambda V_{nk}(t)|^2\,, \tag{163}$$

B:

$$i\hbar\frac{d}{dt}a_n(t) = \sum_k a_k(t)\lambda V_{nk}(t)\,, \tag{164}$$

C:

$$i\hbar\frac{d}{dt}a_n(t) = \sum_k a_k(t)e^{i\omega_{nk}t}\lambda V_{nk}(t)\,, \tag{165}$$

D:

$$i\hbar\frac{d}{dt}a_n(t) = \sum_k a_k(t)e^{i\omega_{nk}t}\,, \tag{166}$$

E:

$$i\hbar\frac{d}{dt}a_n(t) = a_n(t)e^{i\omega_{nn}t}\lambda V_{nn}(t)\,, \tag{167}$$

where $\hbar\omega_{nk} = E_n - E_k$ and $\lambda V_{nk}(t) = \langle\psi_n|\lambda\hat{V}(t)|\psi_k\rangle$.

Solution: C

The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \left(\hat{H}_0 + \lambda\hat{V}(t)\right)|\psi\rangle\,. \tag{168}$$

By inserting the expansion of Eq. (162) and using the properties of the unperturbed eigenstates of Eq. (159), we find the time-dependent equation

$$\sum_k i\hbar\frac{da_k(t)}{dt}e^{-iE_kt/\hbar}|\psi_k\rangle = \sum_k \lambda\hat{V}(t)a_k(t)e^{-iE_kt/\hbar}|\psi_k\rangle\,. \tag{169}$$

We take the inner product of Eq. (169) with respect to $\langle\psi_n^{(0)}(t)| = e^{iE_nt/\hbar}\langle\psi_n|$ and find

$$i\hbar\frac{d}{dt}a_n(t) = \sum_k a_k(t)e^{i\omega_{nk}t}\lambda V_{nk}(t)\,, \tag{170}$$

where $\hbar\omega_{nk} = E_n - E_k$ and $\lambda V_{nk}(t) = \langle\psi_n|\lambda\hat{V}(t)|\psi_k\rangle$.

17. **Fermi's Golden Rule**

Fermi's Golden Rule describes

A: the transition probability,

B: the transition probability per unit time,

C: the transition amplitude ,

D: the transition amplitude per unit time ,

E: the Fermi exclusion principle.

Solution: B

Fermi's golden rule describes the transition probability per unit time.

18. **Scattering Theory - Differential Cross Section**

The differential cross section $d\sigma/d\Omega$ is defined as

A:
$$\frac{\#\text{ particles scattered into } d\Omega \text{ per unit time}}{d\Omega \cdot (\#\text{ incoming particles per area and time})}\,, \tag{171}$$

B:
$$\frac{\#\text{ particles scattered into } d\Omega \text{ per unit time}}{d\Omega \cdot (\#\text{ incoming particles})}\,, \tag{172}$$

C:
$$\frac{\#\text{ particles scattered into } d\Omega \text{ per unit time}}{d\Omega \cdot (\#\text{ incoming particles per time})}\,, \tag{173}$$

D:
$$\frac{\#\text{ particles scattered into } d\Omega \text{ per unit time}}{(\#\text{ incoming particles per time})}\,, \tag{174}$$

E:
$$\frac{\#\text{ particles scattered into } d\Omega}{d\Omega \cdot (\#\text{ incoming particles per area and time})}\,, \tag{175}$$

where # means "number of".

Solution: A

The differential cross section $d\sigma/d\Omega$ is defined as

$$\frac{d\sigma}{d\Omega} = \frac{\text{number of particles scattered into } d\Omega \text{ per unit time}}{d\Omega \cdot (\text{number of incoming particles per area and time})}\,. \tag{176}$$

19. **Scattering Theory - Asymptotic Form of the Wave function**

We consider scattering at a potential that is localized at $\mathbf{r} = 0$ and has a finite extent. An incoming wave of the form

$$\psi_{\mathbf{in}} = Ce^{i\mathbf{k}\cdot\mathbf{r}} \tag{177}$$

travels towards the scattering center, where $\mathbf{k}$ is the incoming wave vector. The coordinate can be expressed in terms of the distance from the origin r and the two angles θ and ϕ, $\mathbf{r} = (r\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The scattered wave $\psi_{\mathbf{scat}}$ far away from the scattering center can be expressed as

A:
$$\psi_{\mathbf{scat}} = Cf(\theta,\phi)e^{ikr}\,, \tag{178}$$

B:
$$\psi_{\mathbf{scat}} = Cf(\theta,\phi)\frac{e^{ikr}}{r}\,, \tag{179}$$

C:
$$\psi_{\mathbf{scat}} = Cf(\theta,\phi)\frac{e^{ikr}}{r^2}\,, \tag{180}$$

D:
$$\psi_{\mathbf{scat}} = Cf(\theta,\phi)\frac{e^{ikr}}{r^3}\,, \tag{181}$$

E:
$$\psi_{\mathbf{scat}} = C\frac{1}{r^2}\,, \tag{182}$$

where $f(\theta,\phi)$ characterizes the scattering strength and the angular dependence of the scattering.

Solution: B

The scattered wave must be proportional to the amplitude of the incoming wave, $\psi_{\mathbf{scat}} \sim C$. Furthermore, the scattered wave must be inversely proportional to the distance from the origin r to conserve the number of particle scattered out of a cross section at distance r, $|\psi_{\mathbf{scat}}|^2 r^2 \sim$ constant. Finally, there is also an angular and strength dependence that is characterized by $f(\theta,\phi)$. In conclusion, the scattered wave is of the form

$$\psi_{\mathbf{scat}} = Cf(\theta,\phi)\frac{e^{ikr}}{r}\,. \tag{183}$$

20. **Gauge Invariance**

The electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{B}(\mathbf{r},t)$ are determined by the electromagnetic vector potential $\mathbf{A}(\mathbf{r},t)$ and the scalar potential $\Phi(\mathbf{r},t)$,

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \tag{184}$$

and

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)\,. \tag{185}$$

The electric and magnetic fields do not specify the electromagnetic vector potential and the scalar potentil uniquely. We can carry out a gauge transformation

$$\mathbf{A}(\mathbf{r},t) \to \mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t) \tag{186}$$

and

$$\Phi(\mathbf{r},t) \to \Phi'(\mathbf{r},t) = \Phi(\mathbf{r},t) - \frac{\partial}{\partial t}\chi(\mathbf{r},t)\,, \tag{187}$$

where $\chi(\mathbf{r},t)$ is an arbitrary differentiable function. This gauge transformation leaves $\mathbf{E}$ and $\mathbf{B}$ invariant.

In the absence of other potentials, the Schrödinger equation for a charged particle in an electromagnetic field described by the field $\mathbf{A}$ and Φ is

$$\left[\frac{(\hat{\mathbf{p}} - q\mathbf{A})^2}{2m} + q\Phi\right]\psi(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t)\,. \tag{188}$$

When we use another gauge described by $\mathbf{A}'$ and Φ', the Schrödinger equation is

$$\left[\frac{(\hat{\mathbf{p}} - q\mathbf{A}')^2}{2m} + q\Phi'\right]\psi'(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r},t)\,. \tag{189}$$

The relation between the wave functions ψ and ψ' is

A:

$$\psi' = \psi\,, \tag{190}$$

B:

$$\psi' = \psi e^{q\chi/\hbar}\,, \tag{191}$$

C:

$$\psi' = \psi e^{-q\chi/\hbar}\,, \tag{192}$$

D:

$$\psi' = \psi e^{iq\chi/\hbar}\,, \tag{193}$$

E:

$$\psi' = \psi e^{-iq\chi/\hbar}\,. \tag{194}$$

Solution: D

Let us try an ansatz

$$\psi' = \psi e^{q\eta}\,, \tag{195}$$

where η is a differentiable spatiotemporal function. Inserting Eq. (195) into Eq. (189), we find

$$\left[\frac{(\hat{\mathbf{p}} - q\mathbf{A}')^2}{2m} + q\Phi'\right]\psi'(\mathbf{r},t) = i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r},t)\,. \tag{196}$$

We now use that

$$i\hbar\frac{\partial}{\partial t}\psi'(,t) = e^{q\eta}\left[i\hbar\frac{\partial}{\partial t}\psi(,t) + i\hbar\frac{\partial\eta}{\partial t}\psi(,t)\right] \tag{197}$$

and

$$\hat{\mathbf{p}}\psi'(,t) = e^{q\eta}\left[\hat{\mathbf{p}}\psi(,t) + \frac{\hbar}{i}\frac{\partial\eta}{\partial\mathbf{r}}\psi(,t)\right] \tag{198}$$

to find that Eq. (196) can be written as

$$e^{q\eta}\left[\frac{(\hat{\mathbf{p}} + \frac{\hbar}{i}\frac{\partial\eta}{\partial\mathbf{r}} - q\mathbf{A}')^2}{2m} + q\Phi'\right]\psi'(\mathbf{r},t) = e^{q\eta}\left[i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r},t) + i\hbar\frac{\partial\eta}{\partial t}\psi(,t)\right]\,. \tag{199}$$

By using Eq. (187) and Eq. (187), we then find the Schrödinger (196) provided

$$\eta(\mathbf{r},t) = \frac{i}{\hbar}\chi(\mathbf{r},t)\,. \tag{200}$$

21. **The Pauli Matrices**

The Pauli matrices $\sigma = (\sigma_x,\, \sigma_y,$ and $\sigma_z)$ are

$$\mathbf{S} = \frac{\boldsymbol{\hbar}}{\mathbf{2}}\boldsymbol{\sigma} \tag{201}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{202}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \tag{203}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{204}$$

The commutator between σ_x and σ_y is then

A:

$$[\sigma_x, \sigma_y] = 2i\sigma_z \,, \tag{205}$$

B:

$$[\sigma_x, \sigma_y] = i\sigma_x \,, \tag{206}$$

C:

$$[\sigma_x, \sigma_y] = 2i\sigma_x \,, \tag{207}$$

D:

$$[\sigma_x, \sigma_y] = 2i\sigma_y \,, \tag{208}$$

E:

$$[\sigma_x, \sigma_y] = 0 \,, \tag{209}$$

Solution: A

We multiply the matrices and find

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z \,. \tag{210}$$

Similarly, we find

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\sigma_z \,. \tag{211}$$

Hence, we have that

$$[\sigma_x, \sigma_y] = 2i\sigma_z \,. \tag{212}$$

22. **Confinement in a Delta-Function Potential**

We consider a particle in a one-dimensional system described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \beta\delta(x) \,, \tag{213}$$

where $\delta(x)$ is the Dirac δ-function and β is a positive constant. The negative bound state energy E_B is then

A:

$$E_b = -\frac{m\beta}{2\hbar^2} \,, \tag{214}$$

B:

$$E_b = -\frac{m\beta}{16\hbar^2}\,, \tag{215}$$

C:

$$E_b = -\frac{m\beta^2}{16\hbar^2}\,, \tag{216}$$

D:

$$E_b = -\frac{m\beta^3}{2\hbar^2}\,, \tag{217}$$

E:

$$E_b = -\frac{m\beta^2}{2\hbar^2}\,. \tag{218}$$

Solution: E

We consider the energy to be negative. The wave function must be normalizable. The solution when $x > 0$ must then be of the form

$$\psi_+(x) = C_+ e^{-\kappa x} \tag{219}$$

where

$$E = -\frac{\hbar^2\kappa^2}{2m}\,. \tag{220}$$

Similarly, the solution when $x < 0$ must be of the form

$$\psi_-(x) = C_- e^{\kappa x}\,. \tag{221}$$

Continuity of the wave function at $x = 0$ requires

$$C_+ = C_- \equiv C\,. \tag{222}$$

Next, we integrate the wave function from $x = 0^-$ to $x = 0^+$

$$\int_{0^-}^{0^+} H\psi(x) = \int_{0^-}^{0^+} E\psi(x)\,, \tag{223}$$

$$-\frac{\hbar^2}{2m}\frac{d\psi(x)}{dx}|_{0^-}^{0^+} - \beta\psi(x=0) = 0\,, \tag{224}$$

$$-\frac{\hbar^2}{2m}(-2C\kappa) - \beta C = 0\,. \tag{225}$$

So that

$$\beta = \frac{\hbar^2}{m}\kappa\,. \tag{226}$$

The bound state energy can then be written as

$$E_b = -\frac{m\beta^2}{2\hbar^2}\,. \tag{227}$$

23. **The Continuity Equation**

The time-dependent Schrödinger equation is

$$\left[\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r})\right]\psi = i\hbar\frac{\partial\psi}{\partial t}\,, \tag{228}$$

where $\hat{\mathbf{p}} = \hbar\nabla/i$ is the momentum operator and $V(\mathbf{r})$ is the scalar potential. From the Schrödinger equation (228), we can derive a continuity equation

$$\frac{\partial n}{\partial t} + \nabla\cdot\mathbf{j} = 0\,, \tag{229}$$

where the particle density is

$$n = \psi(\mathbf{r},t)^*\psi(\mathbf{r},t) \tag{230}$$

and the current density is

A:

$$\mathbf{j} = \left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right]\,, \tag{231}$$

B:

$$\mathbf{j} = \psi^*\nabla\psi\,, \tag{232}$$

C:

$$\mathbf{j} = \frac{\hbar}{2mi}\left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right]\,, \tag{233}$$

D:

$$\mathbf{j} = \frac{\hbar}{2mi}\left[\psi^*\nabla\psi + (\nabla\psi^*)\psi\right]\,, \tag{234}$$

E:

$$\mathbf{j} = \nabla\cdot\frac{\hbar}{2mi}\left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right]\,. \tag{235}$$

Solution: C

We can use the Schrödinger equation (228) to evaluate the rate of change of the particle density:

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial t}\left(\psi^*\psi\right) = \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\,, \tag{236}$$

$$= -\frac{\psi}{i\hbar}\left[\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r})\right]\psi^* + \frac{\psi^*}{i\hbar}\left[\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r})\right]\psi\,, \tag{237}$$

$$= -\frac{\hbar}{2mi}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right]\,, \tag{238}$$

$$= \nabla\left(-\frac{\hbar}{2mi}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]\right) \tag{239}$$

which we can write as the continuity equation (229) with the current density

$$\mathbf{j} = \frac{\hbar}{2mi} \left[\psi^* \nabla \psi - (\nabla \psi^*) \psi \right] . \tag{240}$$

24. **Scattering in One Dimension**

We consider scattering in one dimension, the x-direction. An incoming electron from the left enters a scattering region when $0 \leq x \leq a$. The electron can be reflected back in the negative x-direction or transmitted to propagate towards the right when $x > a$. We assume the potential vanishes when $x < 0$ and $x > a$. In these areas, the particle propagates freely and the relation between the wavevector k and the energy E is $k = (2mE/\hbar^2)^{1/2}$. We assume the energy E is positive. The correct form of the wave function when $x < 0$ for this scattering problem is then

A: $\psi(x) = e^{ikx} + re^{-ikx}$,

B: $\psi(x) = e^{kx} + re^{-kx}$,

C: $\psi(x) = e^{ikx} + re^{-kx}$,

D: $\psi(x) = e^{kx} + re^{-ikx}$,

E: $\psi(x) = re^{-ikx}$.

Solution: A

In the left region, the wave function should describe an incoming wave that propagates to the right and a reflection wave that propagates to the left. The time-dependent wave function is $\Psi(x,t) = \psi(x) \exp iEt/\hbar$ and we then see that only option A describes an incoming wave that propagates to the right and a reflected wave that propagates to the left.

25. **Wave Packet - Phase and Group Velocity**

A photon is described via a wave-packet

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk \phi(k) \exp i(kx - \omega(k)t) , \tag{241}$$

where the frequency is $\omega(k) = ck$. It is assumed that the expansion coefficients $\phi(k)$ has a large and narrow peak with a finite width at $k = k_0$. What is the relation between the phase velocity and the group velocity for this photon?

A: $v_f/v_g = 2$.

B: $v_f/v_g = 1/2$.

C: $v_f/v_g = 3$.

D: $v_f/v_g = 1$.

E: $v_f/v_g = \omega(k)/k$.

Solution: D

The phase velocity is

$$v_f = \frac{\omega(k)}{k} = c\,. \tag{242}$$

The group velocity is

$$v_g = \frac{d\omega(k)}{dk} = c\,. \tag{243}$$

The relation between the phase velocity and the group velocity is therefore,

$$v_f/v_g = 1\,. \tag{244}$$

26. **Density of States**

We consider a two-dimensional system of a finite area L^2. Our focus is on the density of states in the limit that $L \to \infty$. We take into account that there is a factor of 2 degeneracy of each state due to the electron spin. The density of states is in this limit

A:

$$g(E) = 1/L^2\,, \tag{245}$$

B:

$$g(E) = \frac{dN}{dE} = \frac{L^2}{2\pi}\frac{2m}{\hbar^2}\,, \tag{246}$$

C:

$$g(E) = E\,, \tag{247}$$

D:

$$g(E) = \frac{L}{\pi}\sqrt{\frac{2m}{\hbar^2}}E^{-1/2}\,, \tag{248}$$

E:

$$g(E) = E^{3/2}\,. \tag{249}$$

Solution: B

Taking into account the spin-degeneracy, the number of states with a wave vector less than k is

$$N = 2\int_0^k 2\pi k' dk' \left(\frac{L}{2\pi}\right)^2 = \frac{(Lk)^2}{2\pi} = \frac{L^2}{2\pi}\frac{2mE}{\hbar^2}\,. \tag{250}$$

The density of states is then

$$g(E) = \frac{dN}{dE} = \frac{L^2}{2\pi}\frac{2m}{\hbar^2}\,, \tag{251}$$

27. **Quantum Probabilities**

The Hilbert space of a system is spanned by two orthonormal states, $|1\rangle$ and $|2\rangle$. The associated eigenenergies are $E_1 = 1$ and $E_2 = 16$, respectively. We assume that the system is in the state

$$|\psi\rangle = \sqrt{\frac{1}{5}}|1\rangle + \sqrt{\frac{4}{5}}|2\rangle\,. \tag{252}$$

The expectation value of the energy, $E_{\rm avg}$, is then

A:

$$E_{\rm avg} = 1\,, \tag{253}$$

B:

$$E_{\rm avg} = 16\,, \tag{254}$$

C:

$$E_{\rm avg} = 12\,, \tag{255}$$

D:

$$E_{\rm avg} = 13\,, \tag{256}$$

E:

$$E_{\rm avg} = 14\,. \tag{257}$$

Solution: D

The expectation value of the energy is

$$E_{\mathbf{avg}} = \frac{1}{5}E_1 + \frac{4}{5}E_2 = 13 \tag{258}$$

28. **Matrix Mechanics**

The Hilbert space of a system is spanned by three orthonormal states, $|1\rangle$, $|2\rangle$, and $|3\rangle$. Using the matrix mechanics formulation, an operator $\hat{A}$ can be represented in this basis as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{259}$$

The possible eigenvalues of the operator $\hat{A}$ are then

A: -1,0,1 ,

B: -i, 0, i ,

C: -1,1 ,

D: -i, i ,

E: -2,0,2 .

Solution: A

The matrix is block-diagonal. We see that one eigenvalue is zero. The remaining upper-left corner 2×2 matrix equals σ_x, the Paul matrix, that has eigenvalues -1 and 1. In total, we therefore have the eigenvalues -1,0,1.

29. **Addition of Angular Momenta**

We consider two non-interacting spinless particles. The first particle is in an eigenstate corresponding to the its total angular momentum operator

$$\hat{\mathbf{L}_1}^2 |l_1, m_1\rangle = \hbar^2 l_1(l_1+1) \tag{260}$$

with $l_1 = 10$ while the second particle is in an eigenstate corresponding to its total angular momentum operator

$$\hat{\mathbf{L}_!}^2 |l_2, m_2\rangle = \hbar^2 l_2(l_2+1)\,. \tag{261}$$

with $l_2 = 5$.

The possible values for the total angular momentum operator

$$\hat{\mathbf{L}}^2 |l, m\rangle = \hbar^2 l(l+1) \tag{262}$$

are then

A: $l = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$,

B: $l = 5, 5.5, 6, 6.5, 7, 7.5, 8, 8.5, 9, 8.5, 10,$
$10.5, 11, 11.5, 12, 12.5, 13, 13.5, 14, 14.5, 15$,

C: $l = 5, 10, 15$,

D: $l = 5, 15$,

E: $l = 0, 5, 10, 15$.

Solution: A

The minimum total angular momentum quantum number is $l_{\min} = |l_1 - l_2| = 5$ and the maximum total angular momentum quantum number is $l_{\min} = l_1 + l_2 = 15$. All possible integral numbers in between $l_{\min}$ and $l_{\max}$ are possible. Hence, the possible values are $l = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$.

30. **Three-dimensional Box**

A particle is confined in a three-dimensional box of a volume $V = L_x L_y L_z$, where L_x is the length in the x-direction, y is the length in the y-direction, and z is the length in the z-direction. The potential vanishes inside the box and is infinitely large outside the box. The energy levels are then

A:
$$E_{n_x,n_y,n_z} = \frac{\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) , \tag{263}$$

B:
$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) , \tag{264}$$

C:
$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) , \tag{265}$$

D:
$$E_{n_x,n_y,n_z} = \frac{\hbar^2}{2m}\left(\frac{n_x}{L_x} + \frac{n_y}{L_y} + \frac{n_z}{L_z}\right) , \tag{266}$$

E:
$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x}{L_x} + \frac{n_y}{L_y} + \frac{n_z}{L_z}\right) , \tag{267}$$

where $n_x = 1, 2, \ldots$, $n_y = 1, 2, \ldots$, and $n_z = 1, 2, \ldots$ are integral numbers.

Solution: B

The particle is confined in a three-dimensional box. Since the potential vanishes inside the box and is infinitely large outside the box, the wave function is separable into in the three directions. It is then sufficient to consider the one-dimensional box first. Inside the box, when $0 \leq x \leq L_x$, the wave function is

$$\psi_x(x) = A\cos kx + B\sin kx\,. \tag{268}$$

The wave function must vanish at the boundary, $\psi(x=0) = 0$ and $\psi(x=L_x) = 0$. This gives that $A = 0$ and $kL_x = n_x\pi$, where $n_x = 1, 2, \ldots$ is an integral number. The energy is then

$$E_x = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2\pi^2}{2m}\frac{n_x^2}{L_x^2}\,. \tag{269}$$

We get similar results for the energies associated with the motion in the y and z direction so that the total energy is

$$E_{n_X,n_y,n_z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)\,. \tag{270}$$

31. **The Average Kinetic Energy and the Fermi Energy**

We consider an ideal gas of spin 1/2 fermions at zero temperature. The system is then spin degenerate and there are 2 spin states associated with each orbital state. It can then be shown that the relation between the number of particles and the Fermi momentum p_F is

$$N_p = 2\frac{V}{(2\pi\hbar)^3}\frac{4\pi}{3}p_F^3\,. \tag{271}$$

The average kinetic energy per fermion $\langle E\rangle$ is then in terms of the Fermi energy E_F

A:

$$\langle E\rangle = \frac{1}{2}E_F\,, \tag{272}$$

B:

$$\langle E\rangle = E_F\,, \tag{273}$$

C:

$$\langle E\rangle = \frac{3}{2}E_F\,, \tag{274}$$

D:

$$\langle E\rangle = \frac{3}{5}E_F\,, \tag{275}$$

E:

$$\langle E\rangle = \frac{2}{3}E_F\,. \tag{276}$$

Solution: D

The total kinetic energy is

$$E_{\rm tot} = \int d^3p 2\frac{V}{(2\pi\hbar)^3}\frac{p^2}{2m}\,, \tag{277}$$

where the factor of 2 is due to spin-degeneracy and the factor $V/(2\pi\hbar)^3$ is the density of states in momentum space. We should integrate up to the Fermi momentum. We then find

$$E_{\rm tot} = 4\pi 2\frac{V}{(2\pi\hbar)^3}\int_0^{p_F} p^2 dp\frac{p^2}{2m}\,, \tag{278}$$

$$= 2\frac{V}{(2\pi\hbar)^3}\frac{4\pi}{3}p_3^3\frac{3}{5}\frac{p_F}{2m}\,, \tag{279}$$

$$= N_p\frac{3}{5}E_F\,. \tag{280}$$

The average kinetic energy per particle is therefore $\langle E\rangle = 3E_F/5$.

32. **Classical Scattering Theory**

We consider the classical motion of a particle that moves towards a hard wall potential in the shape of a sphere with radius r. The potential is infinitely large inside the sphere and vanishes outside the sphere.

The classical total cross section σ associated with the scattering off the hard wall potential is then

A:

$$\sigma = \frac{4\pi}{3}r^2\,, \tag{281}$$

B:

$$\sigma = \frac{4\pi}{3}r^3\,, \tag{282}$$

C:

$$\sigma = r^2\,, \tag{283}$$

D:

$$\sigma = 2\pi r\,, \tag{284}$$

E:

$$\sigma = \pi r^2\,. \tag{285}$$

Solution: E

When a particle scatters off a hard wall potential, the classical total cross section equals the cross section of the scattering center, e.g.

$$\sigma = \pi r^2\,. \tag{286}$$

33. **One-Dimensional Harmonic Oscillator and Time-Dependence**

The Hamiltonian of a one-dimensional harmonic oscillator can be written in terms of the number operator $\hat{n}$ as

$$\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)\,, \tag{287}$$

where ω is the classical oscillation frequency. The time-dependent Schrödinger equation is

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle\,. \tag{288}$$

The possible energy eigenstates of the time-dependent Schrödinger equation (288) are

$$|n(t)\rangle = e^{iE_n t/\hbar}|n\rangle\,, \tag{289}$$

where

$$\hat{n}|n\rangle = n|n\rangle \tag{290}$$

and $E_n = (n + 1/2)\hbar\omega$.

A system is at $t = 0$ in a state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + |2\rangle + |3\rangle\right]\,. \tag{291}$$

At $t = t_1 \equiv 2\pi\hbar/E_1$, the system is then in the state

A:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + e^{i2\pi/3}|2\rangle + e^{i4\pi/3}|3\rangle\right]\,, \tag{292}$$

B:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + e^{i4\pi/3}|2\rangle + e^{i2\pi/3}|3\rangle\right] , \tag{293}$$

C:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + |2\rangle + |3\rangle\right] , \tag{294}$$

D:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle - |2\rangle + |3\rangle\right] , \tag{295}$$

E:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + e^{i1\pi/3}|2\rangle + e^{i2\pi/3}|3\rangle\right] . \tag{296}$$

Solution: B

The state evolves as

$$|\psi(t=0)\rangle = \frac{1}{3}\left[e^{iE_1 t/\hbar}|1\rangle + e^{iE_2 t/\hbar}|2\rangle + e^{iE_3 t/\hbar}|3\rangle\right] . \tag{297}$$

At $t = 2\pi\hbar/E_1$, we therefore have

$$|\psi(t=2\pi\hbar/E_1)\rangle = \frac{1}{\sqrt{3}}\left[e^{i2\pi}|1\rangle + e^{i2\pi E_2/E_1}|2\rangle + e^{i2\pi E_3 t/E_1}|3\rangle\right] , \tag{298}$$

$$= \frac{1}{\sqrt{3}}\left[e^{i2\pi}|1\rangle + e^{i2\pi 5/3}|2\rangle + e^{i2\pi 7/3}|3\rangle\right] , \tag{299}$$

$$= \frac{1}{\sqrt{3}}\left[|1\rangle + e^{i4\pi/3}|2\rangle + e^{i2\pi/3}|3\rangle\right] . \tag{300}$$

34. **Energy Uncertainty**

A system is described by a time-independent Hamiltonian $\hat{H}$. A particle is in a state

$$|n(t)\rangle = e^{iE_n t/\hbar}|n\rangle \tag{301}$$

where

$$\hat{H}|n\rangle = E_n|n\rangle . \tag{302}$$

The uncertainty of the energy in the state

$$\Delta E_n = \left[\langle n(t)|\hat{H}^2|n(t)\rangle - \Big(\langle n(t)|\hat{H}|n(t)\rangle\Big)^2\right]^{1/2} \tag{303}$$

is then

A:

$$\Delta E_n = E_n/5\,, \tag{304}$$

B:

$$\Delta E_n = E_n/3\,, \tag{305}$$

C:

$$\Delta E_n = E_n/4\,, \tag{306}$$

D:

$$\Delta E_n = 0\,, \tag{307}$$

E:

$$\Delta E_n = E_n/2\,, \tag{308}$$

Solution: D

The system is in an energy eigenstate. There is then no uncertainty of the energy, $\Delta E_n = 0$.

35. **Matrix Mechanics - Hamiltonian**

In the matrix mechanics notation, a system is described by the Hamiltonian $H = \hbar\omega X$, where

$$X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,. \tag{309}$$

What are the possible energy eigenvalues of the system?

A: $\hbar\omega$ and $-\hbar\omega$,

B: $2\hbar\omega$ and $-2\hbar\omega$,

C: $3\hbar\omega$ and $-3\hbar\omega$,

D: $4\hbar\omega$ and $-4\hbar\omega$,

E: $2i\hbar\omega$ and $-2i\hbar\omega$.

Solution: A

The eigenvalues of the matrix X are -1 and 1. The energy eigenvalues are therefore $-\hbar\omega$ and $\hbar\omega$.

Typos in exam 2016

Problem 6: The value of the Planck's constant was incorrect.

Problem 11: While the Hamiltonian in terms of the number operator was correct, there was a missing adjoint operation in the Hamiltonian in terms of the ladder operators.

Problem 13: The last of the Pauli matrices should be the z-component, and not the x-component.

Problem 20: There was a missing differential time factor in the definition of the electric field.

Problem 21: The last of the Pauli matrices should be the z-component, and not the x-component.

Problem 23: The particle density also depends on time, as shown in the continuity equation.

Problem 27: The specified state should be linear superposition of the two states, not just of the first states.

Problem 28: There was a typo in option A. It should have read -1,0,1. As a result, no listed options were correct.

Problem 30: There were typos in all the options. The quantum number n and the length L should have had the subscript z in all the last terms.