

NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET  
 Institutt for fysikk

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**EKSAMEN I**  
**TFY4250 ATOM- OG MOLEKYLFYSIKK og**  
**FY2045 KVANTEFYSIKK**

Lørdag 9. desember 2006

kl. 09.00 - 13.00

Tillatte hjelpemidler: Godkjent kalkulator

Rottmann: Matematisk formelsamling

Øgrim & Lian: Størrelser og enheter i fysikk og teknikk, eller

Lian og Angell: Fysiske størrelser og enheter

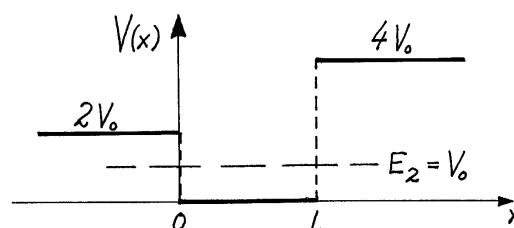
The questions are given in English on page 1 – 5, and then in Norwegian on page 5 – 8.

A sheet with expressions and formulae is attached (page 9)

Sensuren faller i januar 2007.

**ENGLISH TEXT**

**Question 1**



A particle with mass  $m$  is moving in a one-dimensional, symmetric potential well,

$$V(x) = \begin{cases} 2V_0 & \text{for } x < 0, \\ 0 & \text{for } 0 < x < L, \\ 4V_0 & \text{for } x > L. \end{cases}$$

In the calculations below, we assume that  $V_0 = \hbar^2/(2ma_0^2)$ . The width  $L$  is chosen such that the first excited state,  $\psi_2(x)$ , has an energy that is exactly one half of the depth  $2V_0$  of the well, that is, such that  $E_2 = V_0$ .

**a.** From the statements above, one understands that this system has at least two bound energy eigenstates (one for each energy eigenvalue). •State without proof how many zeroes (nodes) the ground state  $\psi_1(x)$  and the first excited state  $\psi_2(x)$  have. •What are the continuity properties of these energy eigenfunctions (when the potential is finite as in this case)?

In the well region, the energy eigenfunction  $\psi_2(x)$  must be sinusoidal and can be written on the form  $\psi_2(x) = A \sin[k_2(x - a)]$  ( $0 < x < L$ ). •Find the wave number  $k_2$  expressed in terms of  $a_0$ . (Remember that  $E_2 = V_0$ .)

**b.** •Show that  $\psi_2(x)$  must have the form  $\psi_2(x) = Ce^{\kappa x}$  for  $x < 0$ , and the form  $\psi_2(x) = C'e^{-\kappa'x}$  for  $x > L$ , and find  $\kappa$  and  $\kappa'$ . •Sketch  $\psi_2(x)$ , and explain why  $a$  must lie in the interval  $0 < a < L$ .

**c.** •Use the continuity conditions for  $x = 0$  to show that  $k_2a$  is equal to  $3\pi/4$ .

•Find also the width  $L$  of the well, expressed in terms of  $a_0$ .

Given:  $\tan(3\pi/4) = \tan(-\pi/4) = -1$ ;  $\tan(5\pi/6) = \tan(-\pi/6) = -1/\sqrt{3}$ .

**d.** For  $E > 2V_0$ , an energy eigenfunction for the potential  $V(x)$  can be written on the form

$$\psi_E = e^{ikx} + re^{-ikx} \quad \text{for } x < 0, \quad \text{with } k = \frac{1}{\hbar}\sqrt{2m(E - 2V_0)}.$$

•Show that this corresponds to a probability density current

$$j(x) = \frac{\hbar k}{m}(1 - |r|^2) \quad (\text{for } x < 0).$$

•Explain why all particles that are coming in from the left towards the well potential  $V(x)$  with energy  $2V_0 < E < 4V_0$  will be reflected. [Hint: Find the form of the energy eigenfunction for  $x > L$ , and calculate the probability density current in this region.]

**e.** For  $E > 4V_0$ , we can choose an energy eigenfunction such that it has the form

$$\psi_E(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{for } x < 0, \\ te^{ik'x} & \text{for } x > L. \end{cases}$$

It can be shown that

$$t = 4e^{-ik'L} \left[ \left(1 + \frac{q}{k}\right)\left(1 + \frac{k'}{q}\right)e^{-iqL} + \left(1 - \frac{q}{k}\right)\left(1 - \frac{k'}{q}\right)e^{iqL} \right]^{-1},$$

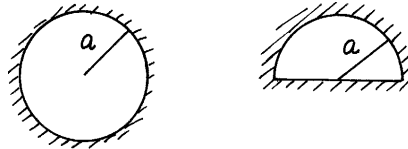
where

$$k = \frac{1}{\hbar}\sqrt{2m(E - 2V_0)}, \quad q = \frac{1}{\hbar}\sqrt{2mE}, \quad k' = \frac{1}{\hbar}\sqrt{2m(E - 4V_0)}.$$

•What is the probability that particles coming in from the left are transmitted when the energy approaches the limit  $4V_0$  from above?

## Question 2

The figure shows two two-dimensional “boxes”, one spherical, the other one in the form of a half sphere, both with radius  $a$ . ( $V = 0$  inside the boxes,  $V = \infty$  outside.)



For the spherical box (on the left), you are informed that a particle with mass  $\mu$  has energy eigenfunctions on the form

$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi); \quad R_{nl}(r) = A_{nl}j_l(\Pi_n^{(l)}r/a); \quad l = 0, 1, \dots; \quad n = 1, 2, \dots$$

The corresponding energy eigenvalues are  $E_{nl} = (\hbar\Pi_n^{(l)})^2/(2\mu a^2)$ . Here,  $\Pi_n^{(l)}$  are the zeroes (nodes) of the spherical Bessel function

$$j_l(z) = z^l \left( -\frac{1}{z} \frac{d}{dz} \right)^l \frac{\sin z}{z}.$$

Some of these zeroes are given in the following table:

	$j_0$	$j_1$	$j_2$	$j_3$
$n = 1$	$\Pi_1^{(0)} = \pi$	$\Pi_1^{(1)} = 4.4934$	$\Pi_1^{(2)} = 5.7635$	$\Pi_1^{(3)} = 6.9879$
$n = 2$	$\Pi_2^{(0)} = \pi \cdot 2$	$\Pi_2^{(1)} = 7.7253$	$\Pi_2^{(2)} = 9.0950$	$\Pi_2^{(3)} = 10.4171$
$n = 3$	$\Pi_3^{(0)} = \pi \cdot 3$	$\Pi_3^{(1)} = 10.9041$	$\Pi_3^{(2)} = 12.3229$	$\Pi_3^{(3)} = 13.6980$

**a.** For  $l = 0$ , the formulae above simplify to

$$R_{n0}(r) = A_{n0} \frac{\sin(\pi nr/a)}{\pi nr/a}; \quad E_{n0} = \frac{(\hbar\pi n)^2}{2\mu a^2}.$$

•Verify these results directly, by the use of the radial equation for the function  $u_{nl}(r) \equiv rR_{nl}(r)$  (see the formula sheet).

**b.** •What are, according to the information given above, the energies of the first and the second excited levels for the spherical box? •Write down the quantum numbers  $(n, l, m)$ , the degree of degeneracy (number of states)  $g$  and the corresponding energy eigenfunctions  $\psi_{nlm}$  for the first excited energy level. (Don't care about normalisation constants, and express  $\psi_{nlm}$  in terms of the relevant Bessel functions and spherical harmonics.) •How many zeroes (nodes) ( $n_r$ ) do the radial functions for the ground state and the first and second levels have in the interval  $0 < r < a$ ?

- c.** • Find the energy and the wave function for the ground state of the “half-sphere” box. [Hint: Take the energy eigenfunctions  $\psi_{nlm}$  for the spherical box as starting point. You may find it convenient to choose a set of axes which makes the  $z$  axis a symmetry axis.]  
 • Find also the energy and the corresponding wave functions for the first excited energy level for the “half-sphere” box.

### Question 3

For a particle with spin  $\frac{1}{2}$ , we may use the spin operator

$$\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma} = \frac{1}{2}\hbar(\hat{\mathbf{e}}_x\sigma_x + \hat{\mathbf{e}}_y\sigma_y + \hat{\mathbf{e}}_z\sigma_z),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the so-called Pauli matrices. the Pauli *spinors*  $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  then are eigenstates of  $S_z = \frac{1}{2}\hbar\sigma_z$ , with the eigenvalues  $\pm\frac{1}{2}\hbar$ . A normalised spin state  $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$  may be characterised by the spin direction,

$$\langle \boldsymbol{\sigma} \rangle = \chi^\dagger \boldsymbol{\sigma} \chi = \hat{\mathbf{e}}_x \Re(2a^*b) + \hat{\mathbf{e}}_y \Im(2a^*b) + \hat{\mathbf{e}}_z (|a|^2 - |b|^2).$$

The matrices  $S_x = \frac{1}{2}\hbar\sigma_x$  satisfy the angular-momentum algebra,

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y.$$

In the present Problem, we assume that the particle is a proton. The inner magnetic moment of this particle then is represented by the operator

$$\boldsymbol{\mu} = \frac{ge}{2m_p} \mathbf{S}, \quad \text{where } g = 5.59.$$

**a.** Suppose that the proton is placed in a homogeneous magnetic field pointing in the negative  $z$  direction,  $\mathbf{B} = -B\hat{\mathbf{e}}_z$ . The behaviour of the spin then is described by the Hamiltonian

$$\hat{H} = -\boldsymbol{\mu} \cdot \mathbf{B} \equiv \boldsymbol{\omega} \cdot \mathbf{S} = \omega S_z,$$

where we have introduced the notation

$$\omega = \frac{geB}{2m_p} \quad \text{and} \quad \boldsymbol{\omega} = \omega \hat{\mathbf{e}}_z = \frac{geB}{2m_p} \hat{\mathbf{e}}_z.$$

- Show that the two Pauli spinors  $\chi_+$  and  $\chi_-$  then are energy eigenstates, and find the two energies,  $E_+$  and  $E_-$ , expressed in terms of the quantities mentioned above. • Find also  $E_+$  and  $E_-$  in electron volts (eV), when  $B = 4$  T(esla). You may use that

$$\mu_N = \frac{e\hbar}{2m_p} = 1 \text{ nuclear magneton} \approx 3.15 \cdot 10^{-8} \text{ eV/T(esla)}.$$

- Find also the wavelength of the photons that appear in transitions between the two states  $\chi_+$  and  $\chi_-$ .

**b.** At  $t = 0$ , a measurement is made of the component  $S_x$  of the proton spin. •What is the measured value, if the spin immediately after the measurement is found in the state

$$\chi(0) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} ?$$

•What are the spin direction  $\langle \boldsymbol{\sigma} \rangle_0$  and the expectation value  $\langle \mathbf{S} \rangle_0 = \frac{1}{2}\hbar \langle \boldsymbol{\sigma} \rangle_0$  of the spin at  $t = 0$ , that is, immediately after the measurement? •Use the general formula for the time development of expectation values,

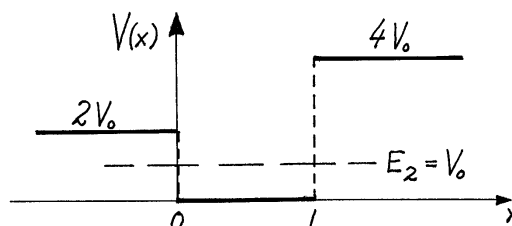
$$\frac{d}{dt} \langle F \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{F}] \rangle,$$

to find the expectation value of  $\langle S_z \rangle_t$  for  $t > 0$ .

**c.** •Find also the expectation values of  $S_x$  and  $S_y$  for  $t > 0$ .

— — —THE END — — —

## Oppgave 1      NORSK TEKST



En partikkel med masse  $m$  befinner seg i et éndimensjonalt, asymmetrisk brønnpotensial

$$V(x) = \begin{cases} 2V_0 & \text{for } x < 0, \\ 0 & \text{for } 0 < x < L, \\ 4V_0 & \text{for } x > L. \end{cases}$$

I beregningene nedenfor antar vi at  $V_0 = \hbar^2/(2ma_0^2)$ . Brønnvidden  $L$  er valgt slik at første eksiterte tilstand,  $\psi_2(x)$ , har en energi som er akkurat lik halvparten av brønnnybden  $2V_0$ , dvs slik at  $E_2 = V_0$ .

**a.** Av opplysningene ovenfor skjønner en at dette systemet har minst to bundne energieigenstater (én for hver energieigenverdi). •Angi uten bevis hvor mange nullpunkter grunntilstanden  $\psi_1(x)$  og 1. eksiterte tilstand  $\psi_2(x)$  har. •Hvilke kontinuitetsegenskaper har disse energieigenfunksjonene (når potensialet er endelig som her)?

I brønnområdet må energieigenfunksjonen  $\psi_2(x)$  være sinusformet og kan skrives på formen  $\psi_2(x) = A \sin[k_2(x - a)]$  ( $0 < x < L$ ). •Finn bølgetallet  $k_2$  uttrykt ved  $a_0$ . (Husk at  $E_2 = V_0$ .)

**b.** •Vis at  $\psi_2(x)$  må ha formen  $\psi_2(x) = Ce^{\kappa x}$  for  $x < 0$ , og formen  $\psi_2(x) = C'e^{-\kappa'x}$  for  $x > L$ , og finn  $\kappa$  og  $\kappa'$ . •Skissér  $\psi_2(x)$ , og forklar hvorfor  $a$  må ligge i intervallet  $0 < a < L$ .

**c.** •Bruk kontinuitetsbetingelsene for  $x = 0$  til å vise at fasebeløpet  $k_2a$  er lik  $3\pi/4$ .

•Finn videre brønnvidden  $L$  uttrykt ved  $a_0$ .

Oppgitt:  $\tan(3\pi/4) = \tan(-\pi/4) = -1$ ;  $\tan(5\pi/6) = \tan(-\pi/6) = -1/\sqrt{3}$ .

**d.** For  $E > 2V_0$  kan en energieigenfunksjon for potensialet  $V(x)$  skrives på formen

$$\psi_E = e^{ikx} + re^{-ikx} \quad \text{for } x < 0, \quad \text{med } k = \frac{1}{\hbar}\sqrt{2m(E - 2V_0)}.$$

•Vis at denne svarer til en sannsynlighetsstrømtetthet

$$j(x) = \frac{\hbar k}{m}(1 - |r|^2) \quad (\text{for } x < 0).$$

•Forklar hvorfor alle partikler som kommer inn fra venstre mot brønnpotensialet  $V(x)$  med energi  $2V_0 < E < 4V_0$  vil bli reflektert. [Hint: Finn formen til energieigenfunksjonen for  $x > L$ , og beregn strømtettheten i dette området.]

**e.** For  $E > 4V_0$  kan vi velge en energieigenfunksjon slik at den har formen

$$\psi_E(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{for } x < 0, \\ te^{ik'x} & \text{for } x > L. \end{cases}$$

Det kan vises at

$$t = 4e^{-ik'L} \left[ \left(1 + \frac{q}{k}\right)\left(1 + \frac{k'}{q}\right)e^{-iqL} + \left(1 - \frac{q}{k}\right)\left(1 - \frac{k'}{q}\right)e^{iqL} \right]^{-1},$$

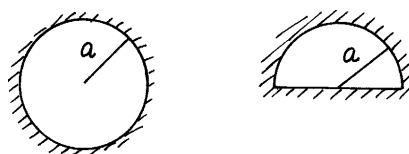
der

$$k = \frac{1}{\hbar}\sqrt{2m(E - 2V_0)}, \quad q = \frac{1}{\hbar}\sqrt{2mE}, \quad k' = \frac{1}{\hbar}\sqrt{2m(E - 4V_0)}.$$

•Hva er sannsynligheten for at partikler som kommer inn fra venstre transmitteres når energien går mot grensen  $4V_0$  ovenfra?

## Oppgave 2

Figuren viser to tredimensjonale “bokser”, den ene kuleformet, den andre i form av en halvkule, begge med radius  $a$ . ( $V = 0$  inne i boksene,  $V = \infty$  utenfor.)



For den kuleformede boksen (til venstre) opplyses det at en partikkel med masse  $\mu$  har energieigenfunksjoner på formen

$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi); \quad R_{nl}(r) = A_{nl} j_l(\Pi_n^{(l)} r/a); \quad l = 0, 1, \dots; \quad n = 1, 2, \dots$$

De tilhørende energieigenverdiene er  $E_{nl} = (\hbar \Pi_n^{(l)})^2 / (2\mu a^2)$ . Her er  $\Pi_n^{(l)}$  nullpunktene i den sfæriske Bessel-funksjonen

$$j_l(z) = z^l \left( -\frac{1}{z} \frac{d}{dz} \right)^l \frac{\sin z}{z}.$$

Noen av disse nullpunktene framgår av følgende oversikt:

	$j_0$	$j_1$	$j_2$	$j_3$
$n = 1$	$\Pi_1^{(0)} = \pi$	$\Pi_1^{(1)} = 4.4934$	$\Pi_1^{(2)} = 5.7635$	$\Pi_1^{(3)} = 6.9879$
$n = 2$	$\Pi_2^{(0)} = \pi \cdot 2$	$\Pi_2^{(1)} = 7.7253$	$\Pi_2^{(2)} = 9.0950$	$\Pi_2^{(3)} = 10.4171$
$n = 3$	$\Pi_3^{(0)} = \pi \cdot 3$	$\Pi_3^{(1)} = 10.9041$	$\Pi_3^{(2)} = 12.3229$	$\Pi_3^{(3)} = 13.6980$

**a.** For  $l = 0$  forenkler formlene ovenfor seg til

$$R_{n0}(r) = A_{n0} \frac{\sin(\pi n r/a)}{\pi n r/a}; \quad E_{n0} = \frac{(\hbar \pi n)^2}{2\mu a^2}.$$

•Verifiser disse resultatene direkte, ved hjelp av radialligningen for funksjonen  $u_{nl}(r) \equiv r R_{nl}(r)$  (se formelarket).

**b.** •Hva er, ifølge opplysningene ovenfor, energiene til første og andre eksiterte nivå for den kuleformede boksen? •Skriv ned kvantetallene  $(n, l, m)$ , degenerasjonsgraden  $g$  og de tilhørende egenfunksjonene  $\psi_{nlm}$  for 1. eksiterte nivå. (Bry deg ikke om normeringskonstanter, og uttrykk  $\psi_{nlm}$  ved de relevante Bessel-funksjonene og sfæriske harmoniske.) •Hvor mange nullpunkter  $(n_r)$  i intervallet  $0 < r < a$  har radialfunksjonene for grunntilstanden og 1. og 2. eksiterte nivå?

**c.** •Finn energien og bølgefunksjonen for grunntilstanden i den halvkuleformede boksen. [Hint: Ta utgangspunkt i energieigenfunksjonene  $\psi_{nlm}$  for den kulesymmetriske boksen. Det kan være greit å velge et aksekors som gjør  $z$ -aksen til symmetriakse.] •Finn også energien og tilhørende bølgefunksjoner for 1. eksiterte nivå for den halvkuleformede boksen.

### Oppgave 3

For en partikkel med spinn  $\frac{1}{2}$  kan en bruke spinnoperatoren

$$\mathbf{S} = \frac{1}{2} \hbar \boldsymbol{\sigma} = \frac{1}{2} \hbar (\hat{\mathbf{e}}_x \sigma_x + \hat{\mathbf{e}}_y \sigma_y + \hat{\mathbf{e}}_z \sigma_z),$$

der

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

er de såkalte Pauli-matrisene. Pauli-spinorene  $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  og  $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  er da egentilstander til  $S_z = \frac{1}{2}\hbar\sigma_z$  med egenverdiene  $\pm\frac{1}{2}\hbar$ . En normert spinntilstand  $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$  kan karakteriseres ved spinnretningen,

$$\langle \boldsymbol{\sigma} \rangle = \chi^\dagger \boldsymbol{\sigma} \chi = \hat{\mathbf{e}}_x \Re(2a^*b) + \hat{\mathbf{e}}_y \Im(2a^*b) + \hat{\mathbf{e}}_z (|a|^2 - |b|^2).$$

Matrisene  $S_x = \frac{1}{2}\hbar\sigma_x$  osv oppfyller dreieimpulsalgebraen,

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y.$$

I denne oppgaven antar vi at partikkelen er et proton. Protonets indre magnetiske moment representeres da av operatoren

$$\boldsymbol{\mu} = \frac{ge}{2m_p} \mathbf{S}, \quad \text{der } g = 5.59.$$

**a.** Anta at protonet befinner seg i et homogent magnetfelt som peker i negativ  $z$ -retning,  $\mathbf{B} = -B\hat{\mathbf{e}}_z$ . Oppførselen til spinnet bestemmes da av Hamilton-operatoren

$$\hat{H} = -\boldsymbol{\mu} \cdot \mathbf{B} \equiv \boldsymbol{\omega} \cdot \mathbf{S} = \omega S_z,$$

der vi har innført betegnelsene

$$\omega = \frac{geB}{2m_p} \quad \text{og} \quad \boldsymbol{\omega} = \omega \hat{\mathbf{e}}_z = \frac{geB}{2m_p} \hat{\mathbf{e}}_z.$$

•Vis at de to Pauli-spinorene  $\chi_+$  og  $\chi_-$  da er energiegentilstander, og finn de to energiene,  $E_+$  og  $E_-$ , uttrykt ved de oppgitte størrelsene. •Finn også  $E_+$  og  $E_-$  i elektronvolt (eV), når  $B = 4$  T(esla). Det oppgis at

$$\mu_N = \frac{e\hbar}{2m_p} = 1 \text{ kjernemagneton} \approx 3.15 \cdot 10^{-8} \text{ eV/T(esla)}.$$

•Finn også bølgelengden til fotonene som er inne i bildet ved overganger mellom de to tilstandene  $\chi_+$  og  $\chi_-$ .

**b.** Ved  $t = 0$  foretas det en måling av komponenten  $S_x$  av spinnet til protonet. •Hva er måleresultatet dersom spinnet umiddelbart etter målingen befinner seg i tilstanden

$$\chi(0) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} ?$$

•Hva er spinnretningen  $\langle \boldsymbol{\sigma} \rangle_0$  og forventningsverdien  $\langle \mathbf{S} \rangle_0 = \frac{1}{2}\hbar \langle \boldsymbol{\sigma} \rangle_0$  av spinnet ved  $t = 0$ , dvs umiddelbart etter målingen? •Bruk den generelle formelen for tidsutviklingen av forventningsverdier,

$$\frac{d}{dt} \langle F \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{F}] \rangle,$$

til å finne forventningsverdien  $\langle S_z \rangle_t$  for  $t > 0$ .

**c.** •Finn også forventningsverdiene av  $S_x$  og  $S_y$  for  $t > 0$ .



## Attachment: Formulae and expressions

Some of the formulae below may turn out to be useful.

### Probability density current

$$j_x(x, t) = \Re \left[ \Psi^*(x, t) \frac{\hbar}{im} \frac{\partial}{\partial x} \Psi(x, t) \right].$$

### Radial equation for spherically symmetric potential $V(r)$

$$\begin{aligned} \psi(r, \theta, \phi) &= R(r)Y_{lm}(\theta, \phi) \equiv \frac{u(r)}{r} Y_{lm}(\theta, \phi); \\ -\frac{\hbar^2}{2\mu} \frac{d^2u}{dr^2} + \left[ V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2} \right] u &= Eu; \quad u(0) = 0. \end{aligned}$$

### Laplace operator and angular-momentum operators in spherical coordinates

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}, \quad \hat{\mathbf{L}}^2 = -\hbar^2 \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right), \\ \hat{L}_x &= \frac{\hbar}{i} \left( -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right), \quad \hat{L}_y = \frac{\hbar}{i} \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right), \quad \hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}, \\ [\hat{\mathbf{L}}^2, \hat{L}_i] &= 0 \quad (i = x, y, z), \quad [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad \text{etc.} \end{aligned}$$

### Spherical harmonics

$$\begin{aligned} \left\{ \begin{array}{c} \hat{\mathbf{L}}^2 \\ \hat{L}_z \end{array} \right\} Y_{lm} &= \left\{ \begin{array}{c} \hbar^2 l(l+1) \\ \hbar m \end{array} \right\} Y_{lm}; \quad \int Y_{l'm'}^* Y_{lm} d\Omega = \delta_{l'l} \delta_{m'm}; \quad \hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}; \\ Y_{00} &= \sqrt{\frac{1}{4\pi}}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}. \\ Y_{20} &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1), \quad Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}, \quad Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}. \end{aligned}$$

### Some physical constants

$$\begin{aligned} a_0 &\equiv \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \frac{1}{\alpha} \frac{\hbar}{m_e c} = 0.529 \times 10^{-10} \text{m}; \quad \alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.036}; \\ c &= 2.998 \times 10^8 \text{m/s}; \quad \hbar = 0.6582 \times 10^{-15} \text{eVs}; \quad m_e = 0.5110 \text{MeV}/c^2. \end{aligned}$$