i General Information

NTNU
Department of Physics

Final Exam - FY2045 Quantum Mechanics I

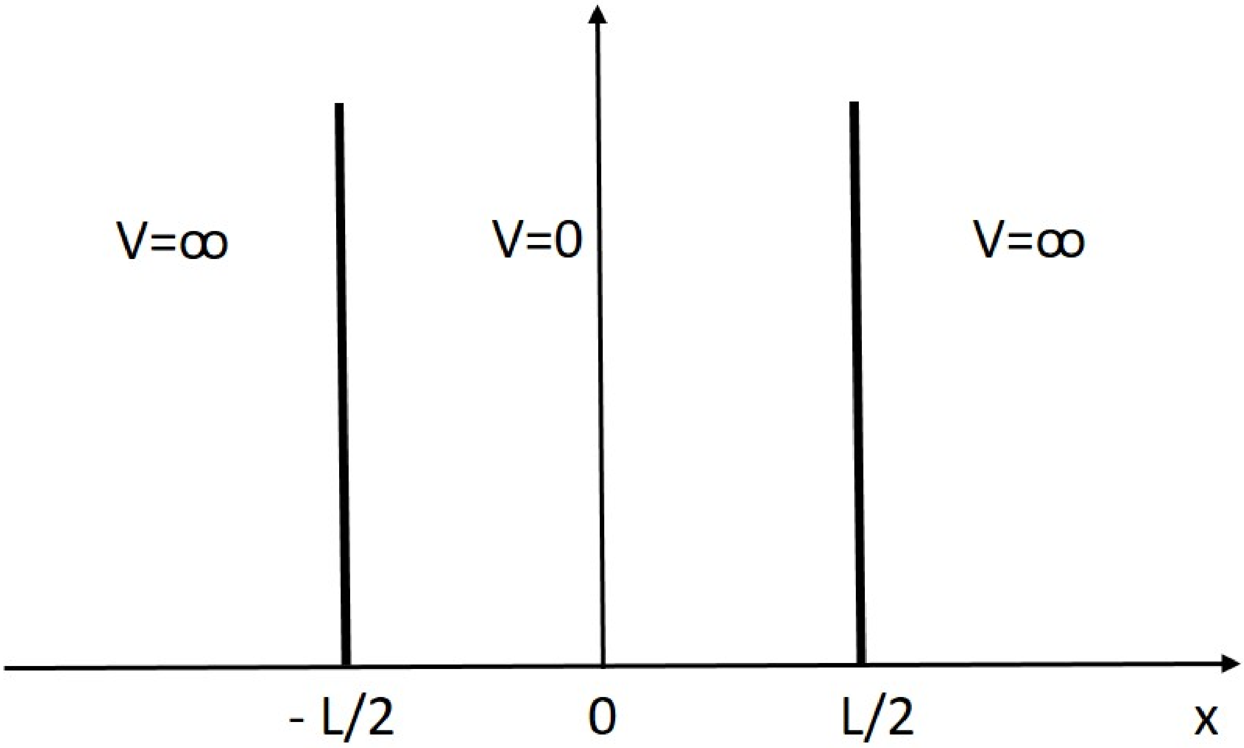
Monday December 03, 2018
Contact during the exam: Mathias Winkler, Tel. 47286769

Exam time: 4 hours
Allowed help: Karl Rottman, Matematisk formelsamling (any language)
Øgrim & Lian, Størrelser og enheter i fysikk og teknikk
Lian & Angell, Fysiske størrelser og enheter
Approved calculator

The exam consists of 4 problems, each counting 25% towards the result. You may answer in English or Norwegian. Section 1 (containing the multiple choice question) is to be answered on the PC. For the remaining problems you are free to write the solutions on the provided sheets.

Feel free to ask if you need help interpreting the questions.

1 Time Independent Perturbation Theory c



For the above shown infinite square well the eigenfunctions are:

$\psi = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right)$ where n = 1, 3, 5,.... and

$\psi' = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ where n = 2, 4, 6,...

Suppose the system is perturbed by a weak potential of the form $V' = \lambda x^3$. The first and second order terms for the energy correction are:

$E_n^{(1)} = \langle n|V'|\, n\rangle$ and $E_n^{(2)} = \sum_{m\neq n} \frac{|\langle n|V'|m\rangle|^2}{E_n - E_m}$

Here we will call the energy correction for the ground state $E_0^{(1)}$, $E_0^{(2)}$. Choose the correct statements below.

Velg ett eller flere alternativer:

- [ ] Only terms that involve states of the form $|m\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ will contribute to $E_0^{(2)}$
- [ ] All first order energy corrections will be zero
- [ ] For $E_0^{(2)}$, the summation above will have an infinite number of terms
- [ ] All second order energy corrections will be zero
- [ ] In time-independent perturbation theory, the power series expansion of the energy $E_n = E_n(\lambda) = E_n^0 + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$ always converges

Maks poeng: 3

2 Singlett/Triplett

Considering a system of two electrons, we can write the total wavefunction as a product of the spatial and spin part, i.e.

$\Psi_{tot} = \Psi_{spatial}\Psi_{spin}$

The spin part can take one of the following forms:

$|\uparrow\uparrow\rangle$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|\downarrow\downarrow\rangle$

corresponding to the so called triplet state and

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

corresponding to the singlett state.

Select all true statements for this system:

Velg ett eller flere alternativer:

- [ ] If the total spin of the system is **S**=0 then the spatial wavefunction has to be symmetric with respect to exchanging the coordinates of the two particles
- [ ] If the total spin of the system is **S**=1, then the spatial wavefunction has to be symmetric with respect to exchanging the coordinates of the two particles
- [ ] Measuring the total spin **S** of the system the possible results are **S** = 0 and **S** = 1
- [ ] Measuring the total spin **S** of the system the possible results are **S** = 1 and **S** = 3
- [ ] If the total spin of the system is **S**=3, then the spatial wavefunction has to be antisymmetric with respect to exchanging the coordinates of the two particles
- [ ] If the total spin of the system is **S**=1 then the spatial wavefunction has to be antisymmetric with respect to exchanging the coordinates of the two particles

Maks poeng: 3

3 Commutator relations 5

Assuming $\hat{O}$ is an operator associated with a physical observable and it obeys the following commutator relationship:

$$\left[\hat{O},\hat{A}\right]=c\hat{A}$$

where $\hat{A}$ is some other operator and c is a non-zero real number. What can we say about the two operators ?

Velg ett eller flere alternativer:

- Eigenstates of $\hat{O}$ are also eigenstates of $\hat{A}$
- $\hat{O}$ must have a finite number of eigenstates
- $\hat{O}$ and $\hat{A}$ are hermitian
- $\hat{O}$ and $\hat{A}$ do not commute
- If $|\psi\rangle$ is an eigenstate of $\hat{O}$, then $\hat{A}|\psi\rangle$ is also an eigenstate of $\hat{O}$

Maks poeng: 2

4 Commutator Relations 2

Given following relations:

$\hat{H}|a\rangle = \epsilon_1|a\rangle \quad \hat{F}|a\rangle = f_1|a\rangle$

$\hat{H}|b\rangle = \epsilon_1|b\rangle \quad \hat{F}|b\rangle = f_2|b\rangle$

$\hat{H}|c\rangle = \epsilon_2|c\rangle \quad \hat{F}|c\rangle = f_2|c\rangle$

$\hat{H}|d\rangle = \epsilon_2|d\rangle \quad \hat{F}|d\rangle = f_2|d\rangle$

Provided that the operators $\hat{H}, \hat{F}$ are hermitian and $\left[\hat{H}, \hat{F}\right] = 0$, which of the following statements is necessarly true ?

Velg ett eller flere alternativer:

- $\langle b|\hat{H}|d\rangle = 0$
- $\langle c|\hat{H}|d\rangle = 0$
- $\langle a|\hat{H}|d\rangle = 0$
- $\langle b|\hat{H}|c\rangle = 0$
- $\langle a|\hat{H}|c\rangle = 0$
- $\langle a|\hat{H}|b\rangle = 0$

Maks poeng: 2.5

5 Time-Dependent Perturbation Theory c

In time-dependent perturbation theory we wish to find an approximate solution for the Schrödinger equation in the case of a time-dependent Hamiltonian:

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$$

Here $\hat{H}_0$ is a time-independent Hamiltonian with known solutions, which form a complete set of orthonormalized eigenstates:

$$|\Psi_n^0(t)\rangle = e^{-iE_n t/\hbar}|\psi_n\rangle \ , \ \ \hat{H}_0|\psi_n\rangle = E_n|\psi_n\rangle.$$

We may then express the solutions of the time-dependent Hamiltonian as a linear combination of the eigenstates of $\hat{H}_0$:

$$|\Psi(t)\rangle = \sum_n a_n(t)|\Psi_n^0(t)\rangle$$

a) Using the time dependent Schrödinger $i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle$, and remembering that $a_n(t) = \langle\Psi_n^0(t)|\Psi(t)\rangle$, show that the time-dependent expansion coefficients $a_n(t)$ are given by the first-order differential equations:

$$i\hbar\frac{d\ a_k(t)}{dt} = \sum_n \langle\Psi_k^0(t)|\hat{V}(t)|\Psi_n^0(t)\rangle a_n(t)$$

b) The above equation may also be written as

$$i\hbar\frac{d\ a_k(t)}{dt} = \sum_n e^{i\omega_{kn}t}V_{kn}(t)a_n(t)$$

where the matrix element $V_{kn}(t)$ and ω_{kn} are defined as:

$$V_{kn}(t) = \langle\psi_k|\hat{V}(t)|\psi_n\rangle$$
$$\omega_{kn} = (E_k - E_n)/\hbar$$

The complete system of equations can then be written as:

$$i\hbar\frac{d}{dt}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & \cdots \\ V_{12}e^{-i\omega_{12}t} & V_{22} & \\ \vdots & & \end{pmatrix}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \end{pmatrix}$$

So far, no approximations have been made.
Introduce the appropriate approximations and (briefly!) explain them, so that above system of equation will simplify. Show that integration then leads to following expression

for the expansion coefficients:

$$a_f(t) \approx \delta_{fi} + \frac{1}{i\hbar}\int_0^t e^{i\omega_{fi}t'} V_{fi}(t')dt'$$

Here the indices i and f refer to initial and final states, respectively.

c) Given the harmonic perturbation:

$$\hat{V}(r,t) = \hat{\mathcal{V}}(r)e^{-i\omega t} + \hat{\mathcal{V}}^\dagger(r)e^{i\omega t}$$

Use the above approximation for $a_f(t)$ to show that there will be a sizeable transition only if the frequency of the perturbation ω is very close to the natural frequency of the system ω_{fi}. (Here you can assume that the matrix elements V_{fi} are very small).

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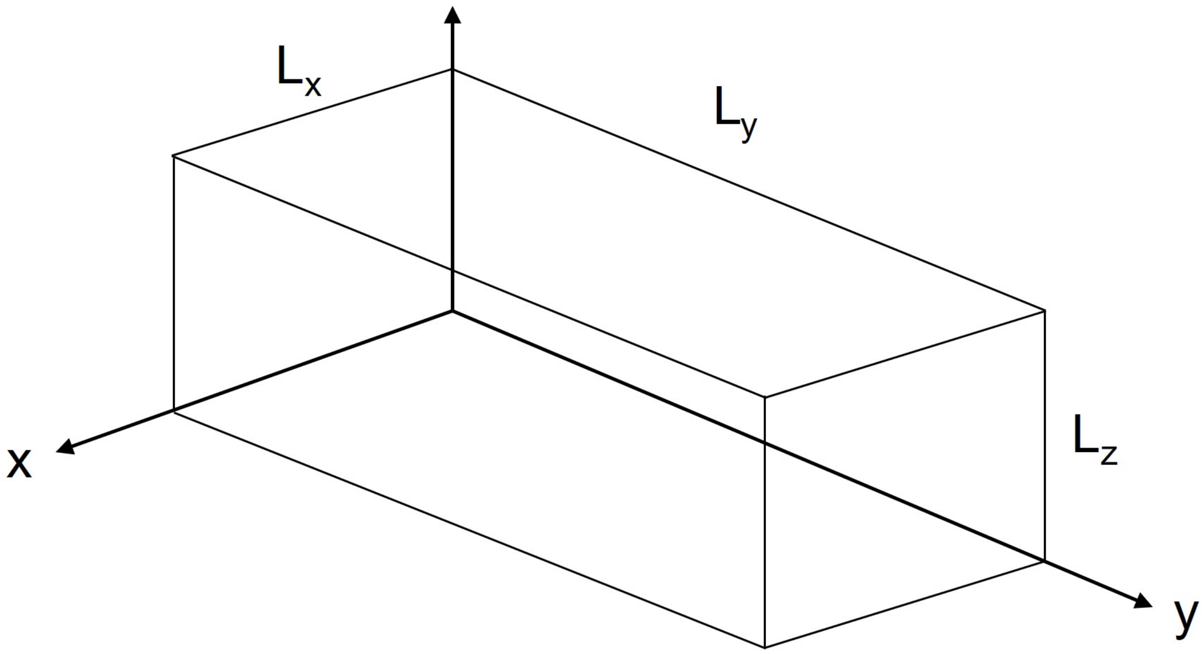
6 Rectangular Box c



The figur shows a rectangular box with edges L_x, L_y, Lz. Assume that the potential is 0 inside the box and infinite outside. The energy eigenfunctions for a particle in the box can be written as:

$$\psi_{n_x n_y n_z}(r) = A \sin \frac{n_x \pi x}{L_x} \sin \frac{n_y \pi y}{L_y} \sin \frac{n_z \pi z}{L_z}$$

a) Find the expression for the energy eigenstates of the system.

b) Assuming that L_x=L_z=L, and L_y = 2L: make a qualitative sketch of the 5 lowest energy level of the system on an Energy scale.

c) Find the pressure on the walls of the box for each direction (x,y,z), assuming that the box contains 7 non-interacting spin 1/2 particles of mass m.

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7 Spin Matrix Representation

In the matrix representation of a spin 1/2 system, the operators for the components of the spin $\hat{\mathbf{S}} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]$ can be written as:

$$\hat{S}_x = \frac{1}{2}\hbar\sigma_x, \hat{S}_y = \frac{1}{2}\hbar\sigma_y, \hat{S}_z = \frac{1}{2}\hbar\sigma_z$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) The raising operator $\hat{S_+}$ can be written as $\hat{S}_x + i\hat{S}_y$. Given the eigenvector of $\hat{S}_z$

$$\chi_{-,z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

apply the raising operator to find the other eigenvector $\chi_{+,z}$ of $\hat{S}_z$. Show that repeated application of the raising operator to $\chi_{+,z}$ yields the zero vector.

b) Find the eigenvalues $S_{\pm,y}$ and eigenvectors $\chi_{\pm,y}$ of the $\hat{S}_y$ operator.

c)

Assume that an electron is placed in an external magnetic field, pointing in the y direction, given by

$$\mathbf{B} = B_0\hat{\mathbf{e}}_y$$

The energy of the interaction between the external field and the spin is dependent on the direction of the magnetic moment (and thus the spin) of the electron, relative to the external field. The Hamiltonian operator can be written as

$$\hat{H} = g_e \frac{-e}{2m_e}\mathbf{B} \cdot \hat{\mathbf{S}}$$

where the constants g_e, e, and m_e are the gyromagnetic factor, the elementary charge and the mass of the electron.

At t=0, the electron is in an eigenstate of $\hat{S}_z$:

$$\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

What is the state of the electron at time $t = \frac{\pi}{2\omega}$, where $\omega = g_e \frac{eB_0}{m_e}$?

Hint: You may find it useful to recall that the differential equation

$$\frac{d^2}{dt^2} a(t) = -\omega^2 a(t)$$

has general solutions

$$a(t) = A_+ e^{i\omega t} + A_- e^{-i\omega t}$$

where A_+ and A_- are constants.

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