5

Time Evolution

This problem can be written to paper.

a) Starting from the time-dependent Schrödinger equation, show that the time-evolution of expectation values is given by:

$$\frac{d\langle\hat{A}\rangle}{dt} = \langle\frac{d\hat{A}}{dt}\rangle + \frac{i}{\hbar}\langle\left[\hat{H}, \hat{A}\right]\rangle$$

(Hint: use that the Hamiltonian is hermitian)

b) Using the relation given above show that for a system described by the general Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x),$$

that the time evolution of the expectation values of the position and the momentum of a particle are given by:

$$m\frac{d}{dt}\langle\hat{x}\rangle = \langle\hat{p}\rangle, \qquad \frac{d}{dt}\langle\hat{p}\rangle = -\langle V'(x)\rangle$$

(Hint: product rule, check the commutator relations in the formula sheet).

c) Using the relations from part b)
i) How do the expectation values for position and momentum evolve in time for a free particle, with initial position x_0 and momentum p_0. How does this compare to classical mechanics ?

ii) How do the expectation values for position and momentum evolve in time for a potential:

$$V = -Fx$$

How does this compare to the classical mechanics ?

6

Time-independent Perturbation Theory

This problem can be written on paper.

Consider a potential, V (x), given by

$$V(x) = \begin{cases} \infty & \text{if} \quad x \leq 0, \\ \lambda\delta(x - L/2) & \text{if} \quad 0 < x \leq L, \\ \infty & \text{if} \quad L < x \end{cases},$$

that is an infinite square well with a delta-function in the middle. In this problem we will study this system using time-independent perturbation theory. We will consider the infinite square well to be the unperturbed potential, where the delta-function is a weak perturbation, which disappears if $\lambda \to 0$.

The perturbed Hamiltonian, $\hat{H} = \hat{H}_0 + \lambda\delta(x - L/2)$ has stationary state solutions, which we will call $|\psi_n\rangle$, with associated energy eigenvalues, . Assume that the energy eigenvalues and eigenstates of the perturbed Hamiltonian can be written as a series expansion in ,

and

a) Writing out the eigenvalue equation for the perturbed Hamiltonian, assume that this equation is satisfied order by order in λ, and show that the equations for the zero and first order terms are:

$$\left(H_0 - E_n^{(0)}\right)\left|\psi_n^{(0)}\right\rangle = 0$$
$$\left(H_0 - E_n^{(0)}\right)\left|\psi_n^{(1)}\right\rangle + \left(\delta(x - L/2) - E_n^{(1)}\right)\left|\psi_n^{(0)}\right\rangle = 0$$

b) Show that the first order correction to the energy eigenvalues is given by

c) Given the that the ground state of the unperturbed Hamiltonian is

find the first order correction to the ground state energy .

d) The first order correction to the energy of the state , and all other states where n is an even number, is 0. Argue on physical grounds why this is the case.

7 Stern-Gerlach Experiment

This Problem can be written on paper.

In the matrix representation of a spin 1/2 system, the operators for the components of the spin $\hat{\mathbf{S}} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]$ can be written as:

$$\hat{S}_x = \frac{1}{2}\hbar\sigma_x, \hat{S}_y = \frac{1}{2}\hbar\sigma_y, \hat{S}_z = \frac{1}{2}\hbar\sigma_z$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

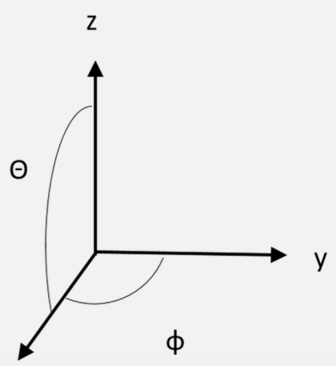
The spin operator for an arbitrary direction, described by the normalized vector $\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$, can be written as:

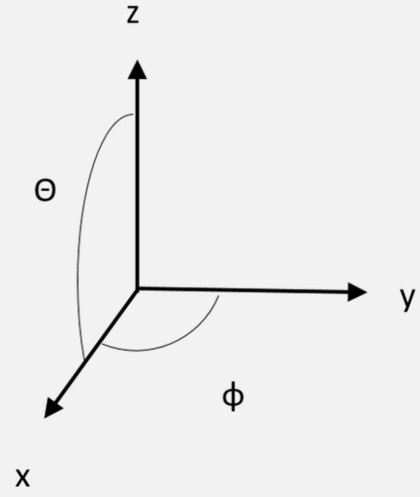
$$\sigma_n = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$$

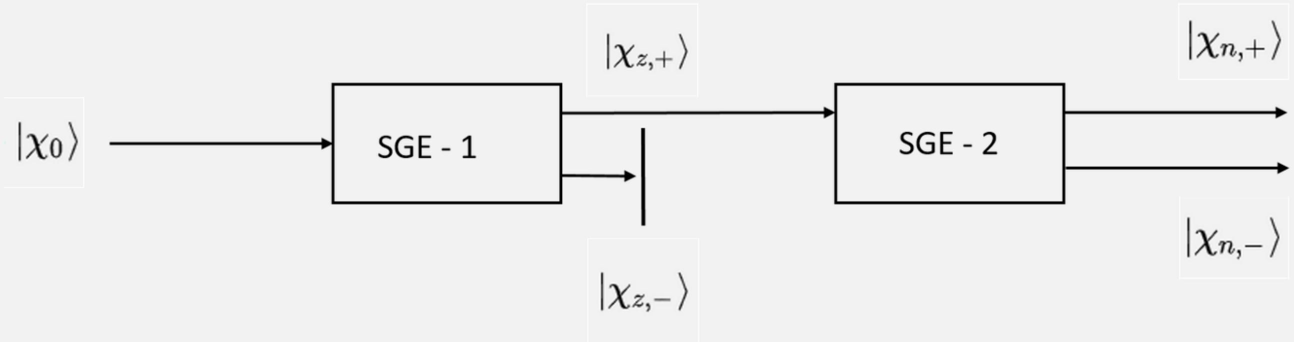
The figure below shows to subsequent Stern-Gerlach experiments (SGE). A beam of electrical neutral particles with spin 1/2 are let through the SGE. Before entering the first SGE the state of the particles can be described by:

$$|\chi_o\rangle = C\begin{pmatrix} 1+2i \\ 4-2i \end{pmatrix}$$

were C is a constant and it is understood that $|\chi_o\rangle$ is written in the basis of eigenvectors of the $\hat{\sigma_z}$ matrix.







a) Normalize $|\chi_0\rangle$ and determine C.

b) The first SGE is oriented along the z direction. Calculate the expectation value of the z-component of the spin measured in the first experiment.

c) The beam of particles that have the spin up state with respect to the z-axis (i.e. $|\chi_{z,+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$) are let through the second SGE. This SGE is tilted by an angle $\Theta = \pi/4$.
What are the probabilities of a particle exiting the apparatus beeing in the spin-up $|\chi_{n,+}\rangle$ and spin-down $|\chi_{n,-}\rangle$ states with respect to the orientation of the second SGE.

d) The eigenvectors of $\hat{\sigma_x}$ are $|\chi_{x,+}\rangle = \sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\chi_{x,-}\rangle = \sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Construct the matrix that represents the raising operator $\hat{\sigma}_+$ for these two vectors, i.e:

$$\hat{\sigma_+}|\chi_{x,-}\rangle = |\chi_{x,+}\rangle$$
$$\hat{\sigma_+}|\chi_{x,+}\rangle = 0$$