NTNU – Trondheim
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NTNU, Department of Physics

Exam FY2045 fall 2021

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09:00–13:00

Permitted examination support material:
No restrictions.

Read carefully. Good luck! Bonne chance! Viel Glück! Veel succes! Lykke til!

Problem 1

Consider the one-dimensional harmonic oscillator whose Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 x^2 \,. \tag{1}$$

We will apply the variational principle using a normalized wavefunction of the form

$$\psi(x) = \sqrt{\alpha}e^{-\alpha|x|} \,, \tag{2}$$

where $\alpha \in \mathbb{R}_+$ is a variational parameter.

a) Calculate the expectation value $\langle T \rangle$ of the kinetic energy in the state $\psi(x)$.

b) Calculate the expectation value $\langle V \rangle$ of the potential energy in the state $\psi(x)$.

c) Extremize the expectation value $\langle H \rangle = \langle T \rangle + \langle V \rangle$ of the total energy with respect to the parameter α and explain that there is a competition between the two terms. Compare the result with the exact ground-state energy of the harmonic oscillator. Any comments?

Problem 2

In this problem we will study the so-called fermionic oscillator. We define two operators b and $b^\dagger$ satisfying the *anticommutator* relations

$$\{b, b\} = \left\{b^\dagger, b^\dagger\right\} = 0 \,, \tag{3}$$

$$\left\{b, b^\dagger\right\} = 1 \,, \tag{4}$$

where the anticommutator of two operators a and b is defined as $\{a, b\} = ab + ba$.

a) Assume that the Hilbert space $\mathcal{H}$ is nonempty such that there is at least one nonzero vector $|\psi\rangle \in \mathcal{H}$. Show that there exists a state $|0\rangle$ such that it is annihilated by b, i.e. $b|0\rangle = 0$.

b) Show that $b^\dagger|0\rangle \neq 0$. Denote the state $b^\dagger|0\rangle$ by $|1\rangle$. If $|0\rangle$ is normalized, what is the length of $|1\rangle$? Find $b|1\rangle$ and $b^\dagger|1\rangle$.

c) Assume that all possible operators in $\mathcal{H}$ are composed of products of b and $b^\dagger$. Show that the Hilbert space is two-dimensional.

d) Is the operator b hermitian? Find the eigenstates and eigenvalues of the operators b and $b^\dagger$.

e) The Hamiltonian of the fermionic oscillator is

$$H = \frac{1}{2}\hbar\omega(b^\dagger b - b b^\dagger) \,, \tag{5}$$

where $\omega > 0$ is a constant. Find the eigenvectors of the Hamiltonian H and the associated energy eigenvalues.

f) One can think of $b^\dagger$ as a creation operator and b as an annihilation operator. Thus from the vacuum state $|0\rangle$, $b^\dagger$ creates the state $|1\rangle$ that contains a single fermion, b creates the vacuum state $|0\rangle$ from $|1\rangle$, i. e. it removes a fermion. Explain why the anticommutator relations above automatically implement the Pauli principle and define an appropriate number operator N.

Problem 3

The proton and the electron that constitute the hydrogen atom are both spin-$\frac{1}{2}$ particles. Their spins are denoted by $\mathbf{S}_e$ and $\mathbf{S}_p$, respectively. The Hamiltonian that describes the interaction between the spins is given by

$$H_{\mathrm{hf}} = \frac{\mu_0 g e^2}{8\pi m_p m_e} \frac{[3(\mathbf{S}_p \cdot \mathbf{e}_r)(\mathbf{S}_e \cdot \mathbf{e}_r) - \mathbf{S}_p \cdot \mathbf{S}_e]}{r^3} + \frac{\mu_0 g e^2}{3 m_p m_e} \mathbf{S}_p \cdot \mathbf{S}_e \, \delta^3(\mathbf{r}) \,, \tag{6}$$

where g is the so-called g-factor. The subscript means hyperfine, which indicates that the effect of H_{hf} is much smaller than the fine structure of the hydrogen spectrum. We will consider H_{hf} a perturbation in the following.

a) It can be shown that

$$I = \int (\mathbf{a} \cdot \mathbf{e}_r)(\mathbf{b} \cdot \mathbf{e}_r) d\Omega = \frac{4\pi}{3} \mathbf{a} \cdot \mathbf{b} \,, \tag{7}$$

where the integral is over the unit sphere, and $\mathbf{a}$, $\mathbf{b}$ are arbitrary vectors in $\mathbb{R}$. Explain why this implies that the expectation value of the term *not* involving the δ-function in Eq. (6) vanishes for $l = 0$ states.

b) Calculate the first-order energy shift of the ground state of hydrogen due to the perturbation Eq. (6). The spatial part of the ground-state wavefunction is $\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$, where a is the Bohr radius.