TFY 4250 / FY 2045 : QUM I - Re-sit exam, August 2015

Solutions

1) a) We demand that

$$1 \stackrel{!}{=} 2|A|^2 \int_0^{\infty} e^{-2amx^2/\hbar}\,dx = 2|A|^2 \frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2 \sqrt{\frac{\pi\hbar}{2am}}$$

$$\Rightarrow \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}} \quad 3$$

b) We know that

$$\frac{\partial\psi}{\partial t} = -ia\psi, \quad \frac{\partial\psi}{\partial x} = -\frac{2amx}{\hbar}\psi, \quad \frac{\partial^2\psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\psi + x\frac{\partial\psi}{\partial x}\right)$$

$$= -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\psi$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \iff V\psi = i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\Rightarrow V\psi = i\hbar(-ia)\psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\psi$$

$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\psi = 2a^2mx^2\psi \Rightarrow \boxed{V(x) = 2ma^2x^2}$$

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c) We find

$$\boxed{\langle x\rangle = \int_{-\infty}^{+\infty} x|\psi|^2\,dx = 0}\ 2, \quad \text{since integrand is odd.}$$

And so

$$\boxed{\langle p\rangle = m\frac{d}{dt}\langle x\rangle = 0}.\ 2$$

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2) First, we find the eigenvalues and eigenvectors of H:

$$\begin{vmatrix} a-E & 0 & b \\ 0 & c-E & 0 \\ b & 0 & a-E \end{vmatrix} = (a-E)(c-E)(a-E) - b^2(c-E)$$

$$= (c-E)\left[(a-E)^2 - b^2\right] = 0 \Rightarrow \boxed{E_1 = c} \text{ or } \boxed{E_2 = a+b} \text{ or } \boxed{E_3 = a-b}$$

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The eigenvectors follow from

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E_n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

After some algebra, we find: (proof also by inspection / educated guess)

$$E_1 = c \;\hat{=}\; |s_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ 1

$$E_2 = a+b \;\hat{=}\; |s_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$ 1

$$E_3 = a-b \;\hat{=}\; |s_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$ 1

We can now write

$$|s(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|s_2\rangle - |s_3\rangle\right),$$ 2

and finally

$$|s(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_2 t/\hbar}|s_2\rangle - e^{-iE_3 t/\hbar}|s_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[e^{-i(a+b)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - e^{-i(a-b)t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right]$$

$$\Rightarrow \boxed{|s(t)\rangle = e^{-iat/\hbar}\begin{pmatrix} -i\cdot\sin(bt/\hbar) \\ 0 \\ \cos(bt/\hbar) \end{pmatrix}}$$ 2

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3) a) We compute

$$\chi^+\chi = |A|^2(9+16) = 25\,|A|^2 \overset{!}{=} 1 \Rightarrow \boxed{A = 1/5}$$ 3

b) Straightforward calculation yields

$$\langle S_x\rangle \overset{\chi^+ S_x \chi =}{=} \frac{1}{25}\frac{\hbar}{2}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}(-12i+12i) = \boxed{0}$$

$$\langle S_y\rangle = \chi^+ S_y \chi = \frac{1}{25}\frac{\hbar}{2}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}(-12-12) = \boxed{-\frac{12}{25}\hbar}$$

$$\langle S_z\rangle = \chi^+ S_z \chi = \frac{1}{25}\frac{\hbar}{2}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}(9-16) = \boxed{-\frac{7}{50}\hbar}$$

c) We have $S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$$\Rightarrow \langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}.$$

Therefore, we find

$$(\sigma_{S_x})^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4} \Rightarrow \boxed{\sigma_{S_x} = \frac{\hbar}{2}}$$

Likewise, we compute

$$(\sigma_{S_y})^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2\hbar^2 \Rightarrow \boxed{\sigma_{S_y} = \frac{7}{50}\hbar}$$

$$(\sigma_{S_z})^2 = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2\hbar^2 \Rightarrow \boxed{\sigma_{S_z} = \frac{12}{25}\hbar}$$

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4) First, we determine the normalization constant

$$1 \stackrel{!}{=} 2|A|^2 \int_0^\infty \frac{dx}{(x^2+b^2)^2} = 2|A|^2 \frac{\pi}{4b^3} \Rightarrow \boxed{A = \sqrt{\frac{2b^3}{\pi}}}$$ 3

Next, we compute $\langle T \rangle$:

$$\langle T \rangle = -\frac{\hbar^2}{2m}|A|^2 \int_{-\infty}^{+\infty} \frac{1}{(x^2+b^2)} \left(\frac{d^2}{dx^2} \frac{1}{x^2+b^2} \right) dx$$

$$= -\frac{\hbar^2}{2m}|A|^2 \int_{-\infty}^{+\infty} \frac{1}{x^2+b^2} \frac{2(3x^2-b^2)}{(x^2+b^2)^3} dx$$

$$= -\frac{\hbar^2}{2m}|A|^2 \int_{-\infty}^{+\infty} \left[2 \cdot \frac{3x^2}{(x^2+b^2)^4} - 2\frac{b^2}{(x^2+b^2)^4} \right] dx$$

int. by parts

$$= -\frac{4\hbar^2 b^3}{\pi m} \left[3\int_0^\infty \frac{dx}{(x^2+b^2)^3} - 4b^2 \int_0^\infty \frac{dx}{(x^2+b^2)^4} \right]$$

$$= -\frac{4\hbar^2 b^3}{\pi m} \left[3\frac{3\pi}{16b^5} - 4b^2 \frac{5\pi}{32b^7} \right] = \boxed{\frac{\hbar^2}{4mb^2}}$$ 2

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Likewise: $$\langle V \rangle = \frac{1}{2}m\omega^2 |A|^2 2\int_0^\infty \frac{x^2}{(x^2+b^2)^2} dx = m\omega^2 \frac{2b^3}{\pi} \frac{\pi}{4b} = \boxed{\frac{1}{2}m\omega^2 b^2}$$

$$\Rightarrow \langle H \rangle = \frac{\hbar^2}{4mb^2} + \frac{1}{2}m\omega^2 b^2 \Rightarrow \frac{d\langle H \rangle}{db} \stackrel{!}{=} 0 \Rightarrow -\frac{\hbar^2}{2mb^3} + m\omega^2 b = 0$$

$$\Rightarrow b^4 = \frac{\hbar^2}{2m^2\omega^2} \Rightarrow b^2 = \frac{1}{\sqrt{2}} \frac{\hbar}{m\omega}$$

$$\Rightarrow \langle H \rangle_{min} = \frac{\hbar^2}{4m} \frac{\sqrt{2}m\omega}{\hbar} + \frac{1}{2}m\omega^2 \frac{1}{\sqrt{2}} \frac{\hbar}{m\omega} = \boxed{\frac{\hbar\omega}{\sqrt{2}}} \left(> \frac{1}{2}\hbar\omega \right) ✓$$

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5) a) By inspection, we find

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ eigenvalue } \boxed{V_0}; \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ eigenvalue } \boxed{V_0};$$

$$X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ eigenvalue } \boxed{2V_0}$$

b) Characteristic equation yields:

$$\det(H-\lambda) = \begin{vmatrix} V_0(1-\varepsilon)-\lambda & 0 & 0 \\ 0 & V_0-\lambda & \varepsilon V_0 \\ 0 & \varepsilon V_0 & 2V_0-\lambda \end{vmatrix} \overset{!}{=} 0$$

$$\Rightarrow \left[V_0(1-\varepsilon)-\lambda\right] \cdot \left[\{(V_0-\lambda)(2V_0-\lambda)\} - (\varepsilon V_0)^2\right] = 0$$

$$\Rightarrow \boxed{\lambda_1 = V_0(1-\varepsilon)}$$

and $(V_0-\lambda)(2V_0-\lambda) - (\varepsilon V_0)^2 = 0$

$$\Rightarrow \lambda^2 - 3V_0\lambda + (2V_0^2 - \varepsilon^2 V_0^2) = 0$$

$$\Rightarrow \lambda = \tfrac{1}{2}\left(3V_0 \pm \sqrt{9V_0^2 - 4(2V_0^2 - \varepsilon^2 V_0^2)}\right) = \frac{V_0}{2}\left[3 \pm \sqrt{1+4\varepsilon^2}\right]$$

$$\approx \frac{V_0}{2}\left[3 \pm (1+2\varepsilon^2)\right]$$

$$\Rightarrow \boxed{\begin{aligned} &\lambda_2 = \frac{V_0}{2}\left(3-\sqrt{1+4\varepsilon^2}\right) \approx V_0(1-\varepsilon^2) \\ \wedge\ &\lambda_3 = \frac{V_0}{2}\left(3+\sqrt{1+4\varepsilon^2}\right) \approx V_0(2+\varepsilon^2) \end{aligned}}$$

/5

c) We have

$$H' = \varepsilon V_0 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow E_3^1 = \langle X_3 | H' | X_3 \rangle = \varepsilon V_0 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{0}. \quad 1$$

($\rightsquigarrow$ no first-order correction)

The second-order correction reads:

$$E_3^2 = \sum_{m=1,2} \frac{|\langle X_m | H' | X_3 \rangle|^2}{E_3^0 - E_m^0}$$

But $\langle X_1 | H' | X_3 \rangle = 0$ (after some algebra)

and $\langle X_2 | H' | X_3 \rangle = \varepsilon V_0 \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \varepsilon V_0$

and $E_3^0 - E_2^0 = 2V_0 - V_0 = V_0$

$$\Rightarrow E_3^2 = \frac{(\varepsilon V_0)^2}{V_0} = \boxed{\varepsilon^2 V_0} \quad 2$$ (second-order correction)

$$\Rightarrow E_3 \approx E_3^0 + E_3^1 + E_3^2 = 2V_0 + 0 + \varepsilon^2 V_0 = \boxed{V_0 (2 + \varepsilon^2)}$$

(the same as we got for λ_3 in b).) 1

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6