NTNU – Trondheim
Norwegian University of
Science and Technology

NTNU, Department of Physics

Solutions Exam FY2045 fall 2020

Lecturer: Professor Jens O. Andersen
Department of Physics, NTNU

Wednesday December 2 2020
15.00-19.00

Problem 1

a) The eigenfunctions satisfy the equation

$$-i\hbar\frac{d\psi_p(x)}{dx} = p\psi_p(x)\;, \tag{1}$$

whose solutions are the usual plane waves

$$\psi_p(x) = \underline{\underline{e^{ipx/\hbar}}}\;. \tag{2}$$

The periodic boundary condition yields

$$\psi_p(0) = \psi_p(L)\;, \tag{3}$$

or

$$1 = e^{ipL/\hbar}\;. \tag{4}$$

The allowed values for the momentum are $p = \frac{2\hbar\pi n}{L}$, where $n = 0 \pm 1, \pm 2, ...$ Integrating $|\psi_p(x)|^2 = 1$ from $x = 0$ to $x = L$ yields L, so the normalized wavefunctions is $\psi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{L}}$.

b) The particle density is

$$\rho = \frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\theta(E_F - E)\,,$$

where $E = \sqrt{m^2c^4 + c^2p^2}$ and $E = \sqrt{m^2c^4 + c^2p_F^2}$. The step function is one for $|p| < |p_F|$ and zero otherwise. The absolute value gives a factor of two and we end up with

$$\rho = \frac{1}{\pi\hbar}\int_0^{p_F} dp = \underline{\underline{\frac{p_F}{\pi\hbar}}}\,. \tag{5}$$

The energy density is

$$\begin{aligned}
\mathcal{E} &= \frac{1}{\hbar}\int\int_{-\infty}^{\infty}\frac{dp}{2\pi}E\theta(E_F - E)\,,\\
&= \frac{1}{\pi\hbar}\int_0^{p_F} dp\sqrt{m^2c^4 + c^2p^2}\\
&= \underline{\underline{\frac{m^2c^3}{2\pi\hbar}\left[x_F\sqrt{x_F^2+1} + \log\left(x_F + \sqrt{x_F^2+1}\right)\right]}}\,,
\end{aligned} \tag{6}$$

where we have used the substitution $x = \sinh u$ and $x = \frac{p}{mc}$. The pressure is calculated in the same manner,

$$\begin{aligned}
P &= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int\frac{dp}{2\pi}(E_F - E)\theta(E_F - E)\\
&= \frac{1}{\pi\hbar}\int_0^{p_F} dp\left[\sqrt{m^2c^4 + c^2p_F^2} - \sqrt{m^2c^4 + c^2p^2}\right]\\
&= \underline{\underline{\frac{m^2c^3}{2\pi\hbar}\left[x_F\sqrt{x_F^2+1} - \log\left(x_F + \sqrt{x_F^2+1}\right)\right]}}\,,
\end{aligned} \tag{7}$$

c) The ultrarelativistic limit is given by $m \to 0$, which yields

$$\mathcal{E} = \underline{\underline{\frac{cp_F^2}{2\pi\hbar}}}\,. \tag{8}$$

$$P = \underline{\underline{\frac{cp_F^2}{2\pi\hbar}}}\,. \tag{9}$$

In the ultrarelativistic case, we have

$$P = \underline{\underline{\mathcal{E}}}\,. \tag{10}$$

Problem 2

a) After integrating over angles, the normalization constant is determined by the integral

$$\begin{aligned} 1 &= |A|^2 4\pi \int_0^\infty r e^{-2\alpha r} r^2 dr \\ &= |A|^2 4\pi \frac{3}{8\alpha^4} \,. \end{aligned} \tag{11}$$

Choosing the phase of the wavefunction to be zero, we find

$$A = \underline{\underline{\sqrt{\frac{2}{3\pi}}\alpha^2}} \,. \tag{12}$$

b) The expectation value of the kinetic energy can be expressed as the integral

$$\langle T \rangle = -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d^3 r \,. \tag{13}$$

For a spherically symmetric function $\psi(r)$, the Laplace operator reduces to

$$\nabla^2 \psi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) \,. \tag{14}$$

This yields

$$\begin{aligned} \langle T \rangle &= -\frac{\hbar^2}{2m} 4\pi \int_0^\infty \psi^* \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) \right] r^2 dr \\ &= \underline{\underline{\frac{\hbar^2}{2m} \frac{\alpha^2}{2}}} \,. \end{aligned} \tag{15}$$

c) The expectation value of the potential energy is

$$\begin{aligned} \langle V \rangle &= -k \int_0^\infty \psi^* \frac{1}{r} \psi r^2 dr \\ &= \underline{\underline{-\frac{4}{3} \frac{\hbar^2}{2m} \frac{\alpha}{a_0}}} \,, \end{aligned} \tag{16}$$

where have used that $k = \frac{e^2}{4\pi\epsilon_0} = \frac{\hbar^2}{m a_0}$.

d) The total energy is the sum of $\langle T \rangle$ and $\langle V \rangle$

$$\langle E \rangle = \frac{\hbar^2}{2m} \left[\frac{1}{2}\alpha^2 - \frac{4}{3} \frac{\alpha}{a_0} \right] \,. \tag{17}$$

The value that minimizes the total energy is found by solving $\frac{d\langle E\rangle}{d\alpha} = 0$, which yields

$$\alpha = \underline{\underline{\frac{4}{3a_0}}} . \tag{18}$$

Since $\frac{d^2\langle E\rangle}{d\alpha^2} > 0$, the value (18) corresponds to a minimum. The energy is then

$$\langle E\rangle_{\min} = \underline{\underline{-\frac{\hbar^2}{2ma_0^2}\frac{8}{9}}} , \tag{19}$$

which is higher than the true value by a factor $\frac{9}{8}$.

Problem 3

a) Since $l = 1$, we have $m_l = 0, \pm 1$. We also have $s_z = \pm\frac{1}{2}$ and so there are $3 \times 2 = 6$ states.

b) Let us consider the commutator $[\hat{\mathbf{J}}^2, \hat{\mathbf{L}}^2]$,

$$[\hat{\mathbf{J}}^2, \hat{\mathbf{L}}^2] = [\hat{\mathbf{L}}^2, \hat{\mathbf{L}}^2] + [\hat{\mathbf{S}}^2, \hat{\mathbf{L}}^2] + 2[\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}, \hat{\mathbf{L}}^2] . \tag{20}$$

The first commutator is trivially zero, while the second vanishes since the two operators act on different subspaces. The last commutator vanishes since $[\hat{\mathbf{L}}, \hat{\mathbf{L}}^2] = 0$ and since the spin and orbital-angular momentum operators act on different subspaces. Similiar arguments can be applied to the other commutators involving angular-momentum operators. Moreover, we find

$$[\hat{\mathbf{J}}^2, \hat{H}] = [\hat{\mathbf{L}}^2, \hat{H}] + [\hat{\mathbf{S}}^2, \hat{H}] + 2[\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}, \hat{H}] . \tag{21}$$

The first commutator vanishes since $[\hat{\mathbf{L}}, \hat{H}] = 0$ and the second vanishes since $\hat{H}$ is independent of spin. The last commutator vanishes by the very same arguments combined. Finally, $[\hat{J}_z, \hat{H}] = 0$ as can be shown in the same manner

c) $l = 1$ and $s = \frac{1}{2}$ yields $j = \frac{1}{2}$ or $j = \frac{3}{2}$. The first case yields $2j + 1 = 2$ states and the second yields $2j + 1 = 4$ states. Thus we have one quadruplet and one doublet and total of 6 states. This is the same as in a) as it must be.

d) The state with $m_l = 1$ and $m_s = \frac{1}{2}$ has $m_j = \frac{3}{2}$ and there is only one way to obtain this. Thus this state is also an eigenstate of $\hat{\mathbf{J}}^2$. We can therefore write

$$|n1\tfrac{3}{2}\tfrac{3}{2}\rangle = \underline{\underline{|n11\tfrac{1}{2}\rangle}} . \tag{22}$$

We can now use the ladder operator $J_- = L_- + S_-$ to construct the other rungs of the ladder with $j = \frac{3}{2}$. Using the general formula $J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}|jm-1\rangle$, we find

$$J_-|n1\tfrac{3}{2}\tfrac{3}{2}\rangle = \hbar\sqrt{3}|n1\tfrac{3}{2}\tfrac{1}{2}\rangle \;, \tag{23}$$

$$(L_- + S_-)|n11\tfrac{1}{2}\rangle = \hbar\sqrt{2}|n10\tfrac{1}{2}\rangle + \hbar|n11-\tfrac{1}{2}\rangle \;. \tag{24}$$

Since the left-hand side of Eqs. (23) and (24) are equal, we obtain

$$|n1\tfrac{3}{2}\tfrac{1}{2}\rangle = \underline{\underline{\sqrt{\frac{2}{3}}|n10\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}|n11-\tfrac{1}{2}\rangle}} \;. \tag{25}$$

By repeated use of J_-, we find

$$|n1\tfrac{3}{2}-\tfrac{1}{2}\rangle = \underline{\underline{\sqrt{\frac{1}{3}}|n1-1\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}|n10-\tfrac{1}{2}\rangle}} \;, \tag{26}$$

$$|n1\tfrac{3}{2}-\tfrac{3}{2}\rangle = \underline{\underline{|n1-1-\tfrac{1}{2}\rangle}} \;. \tag{27}$$

The second ladder has $j = \frac{1}{2}$ and the top rung has $m_j = \frac{1}{2}$ This state must be orthogonal to the state

$$|n1\tfrac{3}{2}\tfrac{1}{2}\rangle \;. \tag{28}$$

This yields

$$|n1\tfrac{1}{2}\tfrac{1}{2}\rangle = \underline{\underline{\sqrt{\frac{1}{3}}|n10\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}|n11-\tfrac{1}{2}\rangle}} \;. \tag{29}$$

The second rung is then found either by using J_- or by requiring orthogonality to the state (26). Either way, we obtain

$$|1\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}\rangle = \underline{\underline{\sqrt{\frac{2}{3}}|n-1\tfrac{1}{2}\rangle - \sqrt{\frac{1}{3}}|n10-\tfrac{1}{2}\rangle}} \tag{30}$$

Problem 4

a) Since $\hat{\mathbf{L}}^2$, $\hat{\mathbf{S}}^2$, $\hat{L}_z$ and $\hat{S}_z$ commute among themselves and commute with the Hamiltonian $\hat{H}$ of an electron moving in an arbitrary spherically symmetric potential $V(r)$, these operators have a simultaneous complete set of eigenvectors.

b) For a given value of l, there are $2l+1$ possible values for m_l. We also have a two-fold degeneracy due to the spin of the electron, giving a total degeneracy of

$$g = \underline{\underline{2(2l+1)}} \,. \tag{31}$$

c) The angular part of the matrix element is proportional to

$$\int Y^*_{lm_l}(\theta,\phi)\cos\theta Y_{l,m'_l}(\theta,\phi)d\Omega \; . \tag{32}$$

The ϕ-dependence of $Y_{lm}(\theta,\phi)$ is $e^{im\phi}$. giving us a factor $e^{i(m'_l-m_l)\phi}$, which vanishes upon integration over ϕ unless $m_l = m'_l$. The parity of $Y_{lm}(\theta,\phi)$ is $(-1)^l$. Since $\cos\theta$ changes sign under parity, $\theta \to \pi - \theta$ and $\phi = \phi + \pi$, the integrand is odd in θ and the integral over θ vanishes. Thus the matrix elements of the perturbation are all zero and there is no first-order Stark effect.

Problem 5

The gravitational pull on a spherical shell of the matter inside the shell is balanced by the force on the same shell arising from the pressure of matter. The pressure could be due to thermal motion or if the temperature is low, the quantum pressure due to the Pauli principle.