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NTNU, Department of Physics

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Problem 1

a) Kepler's laws are

(1) The orbit of a planet is an ellipse with the Sun at one of the two foci, see Fig. 1

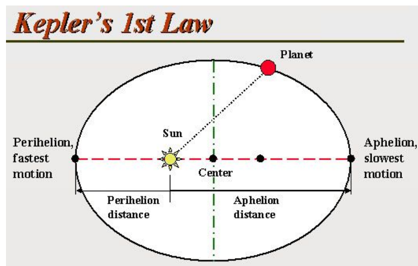

Figure 1: Kepler's first law.

(2) A line segment joining a planet and the Sun sweeps out equal areas during equal time intervals, see Fig. 2. This statement can be written as

$$\frac{dA}{dt} = \underline{\underline{\frac{L}{2m}}} \,. \tag{1}$$

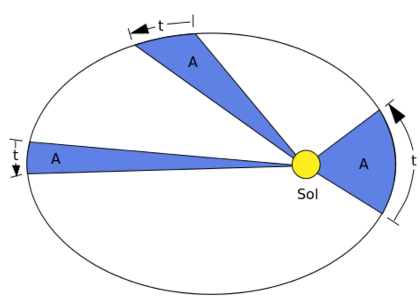

Figure 2: Kepler's second law.

(3) The square of the orbital period P of a planet is proportional to the cube of the semi-major axis a of its orbit,

$$P^2 = \underline{\underline{\frac{4\pi^2 a^3}{GM}}} \,. \tag{2}$$

b) The speed v is given by

$$v = |\mathbf{v}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \, . \tag{3}$$

Taking the derivative of Eq. (1) wrt time yields

$$\begin{aligned} \dot{r} &= \frac{-er_0 \sin\theta}{(1-e\cos\theta)^2}\dot{\theta} \\ &= -\frac{1}{r_0} r^2 e \sin\theta\dot{\theta} \\ &= -\frac{L}{mr_0} e \sin\theta \, , \end{aligned} \tag{4}$$

where we have used the hint $\frac{L}{m} = r^2\dot{\theta}$. The hint also gives

$$\begin{aligned} r\dot{\theta} &= \frac{L}{mr} \\ &= \frac{L}{mr_0}(1-e\cos\theta) \, . \end{aligned} \tag{5}$$

Eqs. (4) and (5) give

$$\begin{aligned} \dot{r}^2 + r^2\dot{\theta}^2 &= \frac{L^2}{m^2 r_0^2}\left[e^2\sin^2\theta + (1-e\cos\theta)^2\right] \\ &= \frac{L^2}{m^2 r_0^2}\left[1+e^2-2e\cos\theta\right] \, , \end{aligned} \tag{6}$$

and therefore

$$v = \underline{\underline{\frac{L}{mr_0}\sqrt{1+e^2-2e\cos\theta}}} \, . \tag{7}$$

The dimensionful constant can be read off and we find

$$A = \underline{\underline{\frac{L}{mr_0}}} = \underline{\underline{\frac{GMm}{L}}} \, . \tag{8}$$

c) The angle α satisfies

$$\begin{aligned} \cos\alpha &= \frac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{v}||\mathbf{k}|} \\ &= -\frac{\mathbf{v}\cdot\mathbf{e}_y}{|\mathbf{v}|} \, . \end{aligned} \tag{9}$$

The scalar product is

$$\begin{aligned} \mathbf{v}\cdot\mathbf{e}_y &= \dot{r}\mathbf{e}_r\cdot\mathbf{e}_y + r\dot{\theta}\mathbf{e}_\theta\cdot\mathbf{e}_y \\ &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \, , \end{aligned} \tag{10}$$

where we have used that

$$\mathbf{e}_r = \cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y \ , \tag{11}$$

$$\mathbf{e}_\theta = -\sin\theta \mathbf{e}_x + \cos\theta \mathbf{e}_y \ . \tag{12}$$

Using the expressions for $\dot{r}$ and $r\dot{\theta}$, we find

$$\begin{aligned} \mathbf{v}\cdot\mathbf{e}_y &= -\frac{L}{mr_0} e\sin^2\theta + \frac{L}{mr_0}(1-e\cos\theta)\cos\theta \\ &= -\frac{L}{mr_0}\left[e-\cos\theta\right] \ . \end{aligned} \tag{13}$$

Substituting this expression and the expression for $|v|$ into Eq. (9), we obtain

$$\cos\alpha = \underline{\underline{\frac{e-\cos\theta}{\sqrt{1+e^2-2e\cos\theta}}}} \ , \tag{14}$$

and we read off B,

$$B = \underline{\underline{1}} \ . \tag{15}$$

d) We simply use the formula given at the end of problem set, where v and α are given in Eqs. (14) and (14). This yields

$$\frac{\omega}{\omega'} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\alpha} = \underline{\underline{\frac{\sqrt{1-\frac{L^2}{m^2 r_0^2 c^2}(1+e^2-2e\cos\theta)}}{1-\frac{L}{mr_0 c}(e-\cos\theta)}}} \ . \tag{16}$$

Problem 2

a) Since the terms inside the first and second paranthesis are dimensionless, M must have the same dimension as r, i.e. length. The singular points are $r=0$ and $r=M$. Based on our knowledge of the singularities in the Schwarzschild metric, we guess that $r=0$ is a physical singularity and that $r=M$ is a coordinate singularity.

b) Consider two events (emission of two photons) at r_A that are separated by the coordinate time Δt_A. The corresponding proper time for an observer at rest at r_A is

$$\Delta\tau_A = \left(1-\frac{M}{r_A}\right)\Delta t_A \ . \tag{17}$$

A similar argument can be used for two events at r_B (reception of two photons) and we find

$$\Delta\tau_B = \left(1-\frac{M}{r_B}\right)\Delta t_B \ . \tag{18}$$

Since the metric is time independent (the path of the second photon in this space-time is identical to the first simply translated in time), we must have $\Delta t_A = \Delta t_B$, which implies

$$\frac{\Delta\tau_A}{\Delta\tau_B} \quad = \quad \frac{\left(1-\frac{M}{r_A}\right)}{\left(1-\frac{M}{r_B}\right)} \; . \tag{19}$$

Since the ratio $\frac{\omega_B}{\omega_A}$ is given by the left-hand side of Eq. (19), we find

$$\omega_B \quad = \quad \underline{\underline{\frac{\left(1-\frac{M}{r_A}\right)}{\left(1-\frac{M}{r_B}\right)}}}\omega_A \; . \tag{20}$$

In the limit $r_A \to r_s$, we find $\omega_B \to 0$ and so we have infinite redshift. This shows that the $r = r_s$ is a horizon, although this is not the Schwarzschild metric.

Problem 3

Friedmann's equations for a homogeneous and isotropic universe are

$$\begin{aligned} 3\frac{\dot{a}^2+k}{a^2} &= 8\pi\rho + \Lambda \; , \\ -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2+k}{a^2} &= 8\pi p - \Lambda \; . \end{aligned}$$

a) $a = a(t)$ is the scale factor, k is the spatial curvature, ρ is the energy density from matter and radiation ($\rho = \rho_m + \rho_r$), p is the pressure, and Λ is the cosmological constant. $\rho_v = \frac{\Lambda}{8\pi}$, where ρ_v is the vacuum energy.

b) Possible values are $k = -1$, $k = 0$, and $k = 1$. $k = -1$ describes a three-dimensional hyperbolic geometry (embedded in four-dimensional Minkowski space). $k = 0$ is three-dimensional Euclidean space, and $k = 1$ is describes a three-dimensional sphere (embedded in four-dimensional Euclidean space).

c) Homogeneous means that all observers are observing the same universe at a given global time t. Isotropic means that an observer is observing the same in all directions. One typically assumes isotropy about every point in the universe (there is no center of the universe).

d) If a is independent of time, $\dot{a} = \ddot{a} = 0$. If $\rho_r = 0$, the pressure p is also vanishing. The second Friedmann then reduces to

$$-\frac{1}{a^2} \quad = \quad -\Lambda \; , \tag{21}$$

which yields the time-independent solution

$$a = \underline{\underline{\sqrt{\frac{1}{\Lambda}}}} \,. \tag{22}$$

The first Friedmann equation reduces to

$$\frac{3}{a^2} = 8\pi\rho_m + \Lambda \,, \tag{23}$$

which yields

$$\rho_m = \underline{\underline{\frac{\Lambda}{4\pi}}} \,. \tag{24}$$

It is interesting to note (although not asked for) that a small perturbation of this universe (a small change in ρ_m, $= \rho_m = \frac{\Lambda}{4\pi} + \delta\rho$) makes it either collapse or expand, i.e. it unstable.

e) There is no gravitational redshift since a is constant. This is at odds with observation.

Problem 4

a) As the gas collapses, the gravitional potential energy decreases. According to the virial theorem, half of this energy is converted into kinetic energy of the gas and is therefore heated. The approximate temperature for the fusion of two protons is 10^7 K.

b) The binding energy is defined by the rest mass energy of the nucleus minus the rest mass energy of its constituents,

$$E_{\rm binding} = \underline{\underline{m_{\rm nuc}c^2 - Nm_Nc^2 - Zm_Pc^2}} \,, \tag{25}$$

where N and Z are the numbers of neutrons and protons in the nucleus.

c) The temperature required for the triple-α process is approximately 10^7 K. The temperature of the interior of all stars satisfies this requirement and therefore the process takes place independently of the mass of the star. This process as well as similar processes produce stable nuclei of C, N, and O. In order to produce heavier nuclei, higher temperatures are required and these are found only in high-mass stars.

Useful formulas

$$\frac{\omega}{c} = \frac{\omega'}{c}\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\alpha} \,. \tag{26}$$