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Problem 1

a) The Doppler shift is given by

$$\frac{\Delta\lambda}{\lambda'} = \frac{v_r}{c} , \tag{1}$$

where $\Delta\lambda = \lambda' - \lambda$ and v_r is the component of the velocity in along the line of sight and λ' is the wavelength when the source is at rest, i.e. $\lambda' = 601.7$ nm.

At $t = 0$ and $t = \frac{1}{2}P$, we see that the two lines are shifted equally. These points therefore correspond to $v_{\rm r} = v_{\rm cm}$. This yields

$$\begin{aligned} v_{\rm cm} &= \frac{\Delta\lambda}{\lambda'}c = \frac{0.1\text{nm}}{601.7\text{nm}} 3 \times 10^8 \text{m/s} \\ &= \underline{\underline{49859\text{m/s}}} . \end{aligned} \tag{2}$$

Since the light is redshifted, the binary is moving away from the observer. Moreover, since the motion is circular, we know that the time $t = \frac{1}{4}P$ corresponds to

$$v_r = v_1 + v_{\rm cm} , \tag{3}$$

$$v_r = v_{\rm cm} - v_2 , \tag{4}$$

for the two stars. Similarly, the time $t = \frac{3}{4}P$ corresponds to

$$v_r = v_1 - v_{\rm cm} , \tag{5}$$

$$v_r = v_2 + v_{\rm cm} , \tag{6}$$

for the two stars. Using the first set of equations, we find [1]

$$v_1 = v_r - v_{\rm cm} , \tag{7}$$

$$v_2 = v_{\rm cm} - v_r \tag{8}$$

Using Eq. (1) to calculate v_r in the two cases, we find

$$\begin{aligned} v_1 &= \frac{\Delta\lambda}{\lambda'}c - v_{\rm cm} \\ &= \frac{0.3\text{nm}}{601.7\text{nm}} 3 \times 10^8 \text{m/s} - 49859\text{m/s} = \underline{\underline{99718\text{m/s}}} , \end{aligned} \tag{9}$$

$$\begin{aligned} v_2 &= \frac{\Delta\lambda}{\lambda'}c - v_{\rm cm} \\ &= 49859\text{m/s} + \frac{0.1\text{nm}}{601.7\text{n}m} 3 \times 10^8 \text{m/s} = \underline{\underline{99718\text{m/s}}} . \end{aligned} \tag{10}$$

The speeds are the same, which should come as a surprise given the symmetry of the measured wavelength. Since the speeds are same, the masses are the same too.

[1] We can equally well use the second set of equations and obtain the same result. The different relative signs are cancelled by the change of sign of $\Delta\lambda$.

b) The center of mass is still moving along the line of sight, it is only the speeds v_1 and v_2 we have to project correctly onto the line of sight. At $t = 0$ and $t = \frac{1}{2}P$, the components of v_1 and v_2 along the line of sight are still vanishin so we will measure the same redshift as in **a**). For the times $t = \frac{1}{4}P$ and $t = \frac{3}{4}P$, we have to multiply v_1 and v_2 by the factor $\sin(90° - i) = 0.8$ since this is the components of the speeds in the direction of the observer. Thus we obtain for $t = \frac{1}{4}P$.

$$v_1 \cos(37°) + v_{\rm cm} = \frac{\Delta\lambda}{\lambda'} c \,, \tag{11}$$

which can be solved with respect to $\Delta\lambda$, giving $\Delta\lambda = 0.26$nm. The remaining shift can be calculated in the same manner and this yields the table

Time/P	$t = 0$	$t = P/4$	$t = P/2$	$t = 3P/4$
Wavelength Gløs	601.8nm	601.96nm	601.8nm	601.64nm
Wavelength Drag	601.8nm	601.64nm	601.8nm	601.96nm

Table 1: Measured wavelengths as a function of time for an inclination angle of $i = 37°$.

Problem 2

a) The binding energy is the mass difference of ^{12}C and its constituents, 6 protons, neutrons, and electrons times c^2.

$$\begin{aligned} \Delta E &= (12 \times 931.5 - 6 \times 939.57 - 6 \times 938.28 - 6 \times 0.511)\text{MeV} \\ &= \underline{\underline{92.166\text{MeV}}} \,. \end{aligned} \tag{12}$$

b) The energy is converted into joules

$$\Delta E = 1.602 \times 10^{-19}\text{J/eV} \times 92.166 \times 10^{6}\text{eV} = 1.47 \times 10^{-11}\text{J} \,. \tag{13}$$

The energy of a photon is $E = \frac{hc}{\lambda}$, giving

$$\lambda = \frac{hc}{\Delta E} = \underline{\underline{1.34 \times 10^{-14}\text{m}}} \,, \tag{14}$$

where we have used $h = 6.6 \times 10^{-34}$Js and $c = 3 \times 10^{8}$m/s. Since visible light is $400 - 700$nm or $(400 - 700) \times 10^{-9}$, we see that the wavelength is a factor 10^7 too short. It is too energetic by the same factor.

Problem 3

a) We need the inverse of the relations given in the lecture notes, which are obtained by making the substitution $v \to -v$ and swapping primed and unprimed coordinates. This yields

$$k^x = \gamma\left(k^{x'} + \frac{v}{c}\frac{\omega'}{c}\right), \qquad (15)$$

$$k^y = k^{y'}, \qquad (16)$$

$$k^z = k^{z'}, \qquad (17)$$

$$\frac{\omega}{c} = \gamma\left(\frac{\omega'}{c} + \frac{v}{c}k^{x'}\right). \qquad (18)$$

Using $(k^0, k^{x'}, k^{y'}, k^{z'}) = (\frac{\omega'}{c}, 0, \frac{\omega'}{c}, 0)$ in S' and Eq. (18), we find

$$\omega = \underline{\underline{\gamma\omega'}}. \qquad (19)$$

Using Eq. (15)–Eq. (18), we find

$$k^x = \gamma\frac{v}{c}\frac{\omega'}{c} = \underline{\underline{\frac{v}{c}\frac{\omega}{c}}}, \qquad (20)$$

$$k^y = \frac{\omega'}{c} = \underline{\underline{\frac{1}{\gamma}\frac{\omega}{c}}}, \qquad (21)$$

$$k^z = \underline{\underline{0}}. \qquad (22)$$

The angle θ si given by

$$\tan\theta = \frac{k^y}{k^x} = \underline{\underline{\frac{1}{\gamma}\frac{c}{v}}}. \qquad (23)$$

b) Since the photon is propagating in the negative x-direction, the four-vector in S' is $(\frac{\omega'}{c}, -\frac{\omega'}{c}, 0, 0)$. The frequency ω is given by Eq. (18)

$$\frac{\omega}{c} = \gamma\left(\frac{\omega'}{c} - \frac{v}{c}\frac{\omega'}{c}\right) = \frac{\omega'}{c}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}. \qquad (24)$$

Inserting the expression for $v(t)$ yields

$$\frac{\omega}{c} = \underline{\underline{\frac{\omega'}{c}\sqrt{\frac{\sqrt{1+\left(\frac{gt}{c}\right)^2} - \frac{gt}{c}}{\sqrt{1+\left(\frac{gt}{c}\right)^2} + \frac{gt}{c}}}}}. \qquad (25)$$

Since $\omega < \omega'$, the photon is redshifted. In the limit $t \to \infty$, $\omega \to 0$, i.e. the photons are infintely redshifted.

Problem 4

a) Setting $r(\theta^*) = R$, we find the angle θ^* that corresponds to the intersection between the circle and the parabola. This yields

$$\cos\theta^* = \underline{\underline{-1 + \frac{r_0}{R}}} \,. \tag{26}$$

b) The angular momentum of the particle of mass m is

$$L = mr^2 \frac{d\theta}{dt} \,, \tag{27}$$

which yields $dt = \frac{m}{L} r^2 d\theta$. The time T it takes from moving from the point $(R, -\theta^*)$ to the point (R, θ^*) on the parabola, i.e the time the comet is inside the circle is

$$\begin{aligned} T &= \int_0^T dt = \frac{m}{L}\int_{-\theta^*}^{\theta^*} r^2 d\theta \\ &= \underline{\underline{\frac{mr_0^2}{L}\int_{-\theta^*}^{\theta^*} \frac{d\theta}{(1+\cos\theta)^2}}} \,. \end{aligned} \tag{28}$$

c) For completeness, we show how to evaluate the integral, although the students were not asked to do this.

We make the substitution $x = \tan\frac{\theta}{2}$. This gives

$$d\theta = \frac{2dx}{1+x^2} \,, \qquad 1 + \cos\theta = 2\cos^2\tfrac{\theta}{2} = \frac{2}{1+x^2} \,. \tag{29}$$

Inserting this into Eq. (28), we can write

$$\begin{aligned} T &= \frac{mr_0^2}{2L}\int_{-x^*}^{x^*} \left[1 + x^2\right] \\ &= \frac{mr_0^2}{L}\left[\tan\tfrac{\theta}{2} + \frac{1}{3}\tan^3\tfrac{\theta}{2}\right] \,. \end{aligned}$$

Finally, using $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$ and $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta$, we find

$$T = \frac{2mr_0^2}{L}\frac{(2+\cos\theta^*)\sin\theta^*}{3\,(1+\cos\theta^*)^2} \,. \tag{30}$$

Using Eq. (26), we can write $\sin\theta = \sqrt{1-\cos^2\theta} = 2\sqrt{\frac{r_0}{R}}\sqrt{1-\frac{r_0}{2R}}$. This yields

$$T = \underline{\underline{\frac{2\sqrt{2}mr_0^2}{3L}\left(\frac{R}{r_0}\right)^{\frac{3}{2}}\left(\frac{r_0}{R}+1\right)\sqrt{1-\frac{r_0}{R}}}} \,. \tag{31}$$

For an elliptical orbit, we have $\frac{L}{m} = \sqrt{GMr_0}$. Inserting this into Eq. (31), one obtains

$$T = \frac{2\sqrt{2}}{3}\sqrt{\frac{R^3}{GM}}\left(\frac{r_0}{R}+1\right)\sqrt{1-\frac{r_0}{2R}} \,. \tag{32}$$

d) The period of Earth's rotation around the Sun is $T_{\text{Earth}} = 2\pi\sqrt{\frac{R^3}{GM}}$. Thus the ratio becomes

$$f = \underline{\underline{\frac{\sqrt{2}}{3\pi}\left(\frac{r_0}{R}+1\right)\sqrt{1-\frac{r_0}{R}}}}\,. \tag{33}$$

To maximize the ratio f, we solve $\frac{df}{dr_0} = 0$, which yields $r_{0,\max} = R$.

$$f_{\max} = \underline{\underline{\frac{2}{3\pi}}}\,. \tag{34}$$

Since T_{Earth} is 365 days, this corresponds to

$$T_{\max} = \frac{2}{3\pi}365 = \underline{\underline{77.5\text{days}}}\,. \tag{35}$$
