NTNU
Faculty of Natural Sciences

NTNU, Department of Physics

Solutions FY2450 Spring 2021

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Problem 1

a) Since the speed v is much smaller than the speed of light, $\frac{v}{c} = \frac{1}{300} \ll 1$, we do not have to take into account special relativity. The relative change in wavelength is

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{1}{300} ,$$

which gives

$$\lambda' = \lambda - \Delta\lambda = \underline{\underline{498.33\text{nm}}} . \quad (1)$$

The wavelength is smaller than in the rest frame, which makes sense since the source is moving toward the observer. The light is *blueshifted*.

b) Since the light from the two stars has zero Doppler shift *at the same time*, the radial velocity must be zero (if it were nonzero, at least light from one of the stars would be Doppler shifted at all times), $v_r = \underline{\underline{0}}$.

The Doppler shift is the same for the two stars. Since the Doppler shift is proportional to the radial velocity, we conclude that their velocities are equal. This implies that the masses are equal, $m_1 = \underline{\underline{m_2}}$. The orbit is therefore a circle, $e = \underline{\underline{0}}$.

c) The Doppler shift is given by

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} . \tag{2}$$

Since the masses are equal, the orbital speeds are equal and given by

$$v_1 = v_2 = c\frac{\Delta\lambda}{\lambda} = \underline{\underline{60000\text{m/s}}} . \tag{3}$$

No, we *cannot* determine the individual masses, except that they are the same. In order to determine the mass, we need to know the period P of the orbit and then use Kepler's third law,

$$M = = m_1 + m_2 = 2m = \frac{P}{2\pi G}(v_1 + v_2)^3 = \underline{\underline{\frac{4P}{\pi G}}}v^3 , \tag{4}$$

where we in the last step have used that $v_1 = v_2 = v$ in this particular case.

d) The inclination angle above is by definition $\frac{\pi}{2}$ and not 0. We must take into account the projection of the speed onto the line of sight, which is done by multiplying by $\cos(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{1}{2}$. Thus the maximum redshift

$$\frac{1}{2} \times 0.1\text{nm} = \underline{\underline{0.05\text{nm}}} . \tag{5}$$

We have earlier calculated the eccentricity e' of a projected ellipse with eccentricity e and angle θ between the planes

$$e'^2 = \underline{\underline{\sin^2\theta + e^2\cos^2\theta}} . \tag{6}$$

Since $e = 0$ and $\theta = \frac{\pi}{2} - \frac{\pi}{6}$, we find

$$e' = \sin\theta = \underline{\underline{\frac{\sqrt{3}}{2}}} . \tag{7}$$

Problem 2

a) Integration yields

$$P(r) = \frac{2\pi}{3}G\rho_c^2 a^2 e^{-r^2/a^2} + C , \tag{8}$$

where C is an integration constant. Using the fact that the pressure vanishes on the surface of the star, $P(R) = 0$ yields

$$P(r) = \underline{\underline{\frac{2\pi}{3}G\rho_c^2 a^2 \left[e^{-r^2/a^2} - e^{-R^2/a^2}\right]}} . \tag{9}$$

b) $m(r)$ is the total mass inside a radius r and $dm = 4\pi r^2 \rho(r) dr$ is the mass in a spherical shell dr added to the system. We add up all such shells by integrating from $r = 0$ to $r = R$. Using integration by parts, the left-hand side is

$$\begin{aligned} 4\pi \int_0^R r^3 \frac{dP}{dr} dr &= 4\pi r^3 P\Big|_0^R - 3\int_0^R 4\pi P(r) r^2 dr \\ &= -3\int P(\mathbf{r}) d^3r \equiv -3V\langle P\rangle \,, \end{aligned} \tag{10}$$

where we have used that the surface term vanishes since either $r = 0$ or $P(R) = 0$, and that the average of a function $f(\mathbf{r})$ is defined as

$$\langle f\rangle = \frac{1}{V}\int f(\mathbf{r}) d^3r \,. \tag{11}$$

c) In order to calculate U we must integrate this expression,

$$\begin{aligned} U &= -12\pi \int_0^R P(r) r^2 dr \\ &= -8\pi^2 G\rho_c^2 a^2 \int_0^R \left[e^{-r^2/a^2} - e^{-R^2/a^2}\right] r^3 dr \\ &= -\frac{2\pi^2}{3} G\rho_c^2 a^2 \left[-2e^{-R^2/a^2} R(3a^2 + 2R^2) + 3a^3\sqrt{\pi}\,\mathrm{erf}(\tfrac{R}{a})\right] \,. \end{aligned} \tag{12}$$

d) We need to expand the error function to order x^5 ($x = R/a$), as indicated

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + ...\right] \,. \tag{13}$$

We also need to expand the exponential to order

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 \,, \tag{14}$$

to get the first nonzero contribution. The order-R and R^3 terms cancel and the first nonzero term is of order R^5. We find

$$U = -\frac{16\pi^2 G\rho_c^2}{15} R^5 = \underline{\underline{-\frac{3}{5}\frac{GM^2}{R}}} \,, \tag{15}$$

where we have defined $M = \frac{4\pi}{3}\rho_c R^3$. This is therefore the same result as we obtain for a uniform mass distribution of radius R and mass density ρ_c.

Problem 3

a) The neutrons in a neutron star satisfy the Pauli principle. This gives rise to a quantum or degeneracy pressure that counteracts the gravitational pull inwards. Hydrostatic equilibrium is obtained by the balance of these two forces. Since the degeneracy pressure is of quantum-mechanical origin and the neutron star is a macroscopic object(!), it is reasonable statement.

b) During the 12h it takes for Earth to complete half a revolution, the Moon has also been moving around its orbit around the Earth. It therefore takes extra time before the same point on Earth is facing the Moon again, which happens to be 26 minutes

c) Heavier elements such as Carbon and Oxygen have a larger Coulomb barrier since the total electric charge is larger. More kinetic energy is required to overcome the barrier and this translates into higher temperatures. Quantum tunnelling lowers the temperature necessary for fusion compared to classical considerations.