Department of Physics

Examination paper for
FY8104 Application of symmetry groups in physics

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Examination time (from-to): 9-13

Permitted examination support material: C

Approved calculator

Rottmann: Matematisk formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of two problems, each containing several subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved.

Under normal circumstances, each subproblem will be given approximately equal weight during grading, except subproblem 2d, which may be given a higher weight.

Some formulas can be found on the pages following the problems.

Language: English

Number of pages (including front page and attachments): 7

Informasjon om trykking av eksamensoppgave

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Problem 1.

SO(3) is the group of (proper) rotations in 3 dimensions, all rotations being around axes passing through the origin. A general rotation $R \in \mathrm{SO}(3)$ can be written $R = R(\boldsymbol{n}, \eta)$, where $\boldsymbol{n}$ is a unit vector pointing in the direction of the rotation axis and η is the rotation angle.

a) What are the conjugacy classes of SO(3)? Briefly justify your answer.

Consider a system with angular momentum operator $\boldsymbol{J} = (J_x, J_y, J_z)$. The states $|j,m\rangle$ are eigenstates of $\boldsymbol{J}^2$ and J_z:

$$\boldsymbol{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \tag{1}$$
$$J_z|j,m\rangle = \hbar m|j,m\rangle, \tag{2}$$

where $j = 0, 1/2, 1, 3/2, \ldots$ and $m = -j, -j+1, \ldots, j$. Under an SO(3) rotation $R = R(\boldsymbol{n}, \eta)$, the state $|j,m\rangle$ transforms like

$$U(R)|j,m\rangle = \sum_{m'=-j}^{j} |j,m'\rangle D^{(j)}_{m'm}(R). \tag{3}$$

where

$$U(R) = \exp(-i\boldsymbol{J}\cdot\boldsymbol{n}\eta/\hbar) \tag{4}$$

and the $D^{(j)}(R)$ are matrices for an irreducible representation $D^{(j)}$ of SO(3).

b) Express $D^{(j)}_{m'm}(R)$ as a matrix element of $U(R)$ in the orthonormal basis $\{|j,m\rangle\}$ for a fixed value of j. Calculate $D^{(j)}_{m'm}(R)$ for a rotation R around the z axis, i.e. $\boldsymbol{n} = \hat{\boldsymbol{z}}$.

c) Show that the character of the representation $D^{(j)}$ is given by

$$\chi^{(j)}(R) = \frac{\sin\left((j+1/2)\eta\right)}{\sin(\eta/2)}. \tag{5}$$

A rank-k spherical tensor operator with (spherical) components $T^{(k)}_q$ ($q = -k, \ldots, k$) transforms under rotations like

$$U(R)T^{(k)}_q U^\dagger(R) = \sum_{q'=-k}^{k} T^{(k)}_{q'} D^{(k)}_{q'q}(R). \tag{6}$$

d) Show that

$$[J_z, T^{(k)}_q] = \hbar q\, T^{(k)}_q. \tag{7}$$

The position operator $\boldsymbol{r}$ and the momentum operator $\boldsymbol{p}$ are rank-1 spherical tensor operators. Their spherical components are:

$$\text{For } \boldsymbol{r}: \quad r_1 = -\frac{1}{\sqrt{2}}(x+iy), \quad r_0 = z, \quad r_{-1} = \frac{1}{\sqrt{2}}(x-iy), \tag{8}$$

$$\text{For } \boldsymbol{p}: \quad p_1 = -\frac{1}{\sqrt{2}}(p_x+ip_y), \quad p_0 = p_z, \quad p_{-1} = \frac{1}{\sqrt{2}}(p_x-ip_y). \tag{9}$$

e) Use the formula

$$T_q^{(k)} = \sum_{q_1,q_2} \langle k_1 k_2; q_1 q_2 | k_1 k_2; kq\rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)} \tag{10}$$

to construct a rank-0 spherical tensor operator from $\boldsymbol{r}$ and $\boldsymbol{p}$. Express it in terms of the cartesian components of $\boldsymbol{r}$ and $\boldsymbol{p}$. Is the form of the final result reasonable? (Below are tables of Clebsch-Gordan coefficients $\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$ for the case $j_1 = j_2 = 1$ and nonnegative m. Coefficients for negative m may be obtained from $\langle j_1 j_2; -m_1, -m_2 | j_1 j_2; j, -m\rangle = (-1)^{j_1+j_2-j}\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$.)

m=2

m_1, m_2 \ j	2
1, 1	1

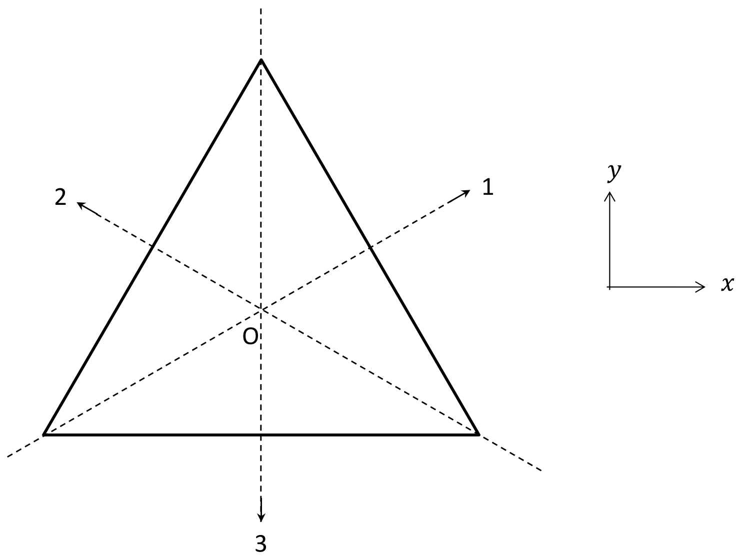
m=1

m_1, m_2 \ j	2	1
1, 0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
0, 1	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

m=0

m_1, m_2 \ j	2	1	0
1, -1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
0, 0	$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$
-1, 1	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

Problem 2.



The point group D_3 is the symmetry group of rotations of an equilateral triangle. In the figure above, the triangle lies in the xy plane, and the z axis points out of the figure and passes through the center of the triangle (the origin O). D_3 contains the following 6 elements: e (the identity element), c (clockwise rotation by angle $2\pi/3$ around the z axis), c^2 (clockwise rotation by angle $4\pi/3$ around the z axis), and b_I (rotation by angle π about the axis OI), where $I = 1, 2, 3$ (the three axes OI are shown as dashed lines in the figure; the axis O3 coincides with the negative y axis).

a) (i) Show that $b_1 c = b_3$. (ii) The multiplication table for D_3 is shown below. Determine the 6 missing entries (you may use information from the multiplication table).

	e	c	c^2	b_1	b_2	b_3
e	e	c	c^2	b_1	b_2	b_3
c	c	c^2	e	b_2	b_3	b_1
c^2	c^2	e	c			
b_1	b_1	b_3	b_2			
b_2	b_2	b_1	b_3	c	e	c^2
b_3	b_3	b_2	b_1	c^2	c	e

b) Find the conjugacy classes of D_3.

c) Find the subgroups of D_3.

d) The character table for D_3 is shown on the next page. Identify the labels denoted by question marks and derive the numbers in the table.

	?	?	?
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	1	-1
$\Gamma^{(3)}$	2	-1	0

e) For each irreducible representation of D_3, identify the kernel and verify that it is a normal subgroup.

In the following we will consider a representation Γ of D_3 that determines how vectors in ordinary 3-dimensional space transform, with the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ as an orthonormal basis. The matrices for the elements c and b_3 are

$$\Gamma(c) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma(b_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{11}$$

f) Determine the representation matrices for the other elements of D_3. (If you know of more than one method, use the simplest one.) All the matrices should have determinant 1; why?

g) Γ is a reducible representation of D_3. Explain how the fact that Γ is reducible can be seen directly from the character table of D_3 (i.e. without doing any calculations). Calculate the decomposition of Γ into irreducible representations of D_3.

h) Argue that if one restricts the irreducible representation $D^{(1)}$ of the group SO(3) to the discrete rotations that are elements of D_3, one obtains a representation of D_3. Show that this representation is equivalent to Γ.

i) Consider an atomic energy level that can be associated with the irreducible representation $D^{(2)}$ of SO(3). Determine how this level splits if the atom is placed in an external field with D_3 symmetry.

Formulas

Some formulas that may or may not be helpful (you should know the meaning of the symbols and possible limitations of validity):

$$f : A \to B \quad \Rightarrow \quad \mathrm{Im}(f) \cong A/\mathrm{Ker}(f)$$

$$O(A)\psi(\boldsymbol{r}) = \psi(A^{-1}\boldsymbol{r})$$

$$O(A)\phi_i = \sum_j \phi_j \Gamma_{ji}(A)$$

$$\sum_A \Gamma_{ik}^{(\alpha)*}(A)\Gamma_{jl}^{(\beta)}(A) = \frac{|G|}{d_\alpha}\delta_{\alpha\beta}\delta_{ij}\delta_{kl}$$

$$\sum_\alpha d_\alpha^2 = |G|$$

$$\sum_A \chi^{(\alpha)*}(A)\chi^{(\beta)}(A) = \sum_c e_c\, \chi_c^{(\alpha)*}\chi_c^{(\beta)} = |G|\delta_{\alpha\beta}$$

$$\sum_\alpha e_c\, \chi_c^{(\alpha)*}\chi_{c'}^{(\alpha)} = |G|\delta_{c,c'}$$

$$a_\alpha = \frac{1}{|G|}\sum_A \chi^{(\alpha)*}(A)\chi(A)$$

$$P_{kl}^{(\alpha)} = \frac{d_\alpha}{|G|}\sum_A \Gamma_{kl}^{(\alpha)*}(A)O(A)$$

$$\chi^{(\alpha\times\beta)}(A) = \chi^{(\alpha)}(A)\chi^{(\beta)}(A)$$

$$\chi^{[\alpha\times\alpha]}(A) = \frac{1}{2}\left[(\chi^{(\alpha)}(A))^2 + \chi^{(\alpha)}(A^2)\right]$$

$$\chi^{\{\alpha\times\alpha\}}(A) = \frac{1}{2}\left[(\chi^{(\alpha)}(A))^2 - \chi^{(\alpha)}(A^2)\right]$$

$$\chi^{(j)}(\eta) = \frac{\sin((j+1/2)\eta)}{\sin(\eta/2)}$$

$$\frac{1}{\pi}\int_0^\pi d\eta\,(1-\cos\eta)\chi^{(l)*}(\eta)\chi^{(l')}(\eta) = \delta_{l,l'}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\eta\,(1-\cos\eta)\chi^{(j)*}(\eta)\chi^{(j')}(\eta) = \delta_{j,j'}$$

$$J_{\pm}|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

$$D^{(j_1)}_{m_1 m_1'}(R) D^{(j_2)}_{m_2 m_2'}(R) = \sum_{j,m,m'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm'\rangle D^{(j)}_{mm'}(R)$$

$$O(A) T^{(\alpha)}_m O^{\dagger}(A) = \sum_{m'} T^{(\alpha)}_{m'} D^{(\alpha)}_{m'm}(A)$$

$$\begin{aligned}
[J_z, T^{(k)}_q] &= \hbar q T^{(k)}_q \\
[J_{\pm}, T^{(k)}_q] &= \hbar\sqrt{(k \mp q)(k \pm q + 1)} T^{(k)}_{q \pm 1}
\end{aligned}$$

$$\langle \alpha', j'm' | T^{(k)}_q | \alpha, jm\rangle = \langle jk; mq | jk; j'm'\rangle \langle \alpha' j' || T^{(k)} || \alpha j\rangle$$

$$T^{(k)}_q = \sum_{q_1, q_2} \langle k_1 k_2; q_1 q_2 | k_1 k_2; kq\rangle X^{(k_1)}_{q_1} Z^{(k_2)}_{q_2}$$