THE NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF PHYSICS

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Examination, course FY8104 Symmetry in physics
Tuesday December 13, 2011
Time: 09.00–13.00

Grades made public: Friday December 30, 2011

Allowed to use: Calculator, mathematical tables.

All subproblems are given the same weight in the grading.

Problem 1:

The four-group has four elements e, a, b, c with the following multiplication table.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Write down its character table.
Can you find a faithful irreducible representation?
Can you find a faithful reducible representation?

Problem 2:

The Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generate a group G with the 16 elements $\pm\mathbf{I}$, $\pm\mathrm{i}\mathbf{I}$, $\pm\boldsymbol{\sigma}_n$, $\pm\mathrm{i}\boldsymbol{\sigma}_n$ for $n = 1, 2, 3$.
$\mathbf{I}$ is the 2×2 identity matrix.

The most important relations defining the multiplication table are the following,

$$\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = \mathbf{I} \,, \qquad \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \mathrm{i}\boldsymbol{\sigma}_3 \,.$$

You are not required to write out the complete multiplication table.

a) What are the orders of the different group elements?

What are the conjugation classes of G?

b) Find subgroups of G. Which subgroups are normal (invariant)?

c) Fill in as much as you can of the character table.

Hints:

- All members of a conjugation class must have the same order.
- In a one dimensional representation the character is the representation.
- A representation of a quotient group G/H, where H is a normal subgroup, is also a representation of G.
- If a group element and its inverse are conjugates, then the character value is real.
- The complex conjugate of a representation is also a representation.

The following orthogonality relations hold for a finite group of order N.

Let $\chi_i^{(\mu)}$ be the character value of the conjugation class i, with N_i elements, in the irreducible representation μ. Then

$$\sum_i N_i (\chi_i^{(\mu)})^* \chi_i^{(\nu)} = N\,\delta_{\mu\nu} \; ,$$

$$\sum_\mu (\chi_i^{(\mu)})^* \chi_j^{(\mu)} = \frac{N}{N_i}\,\delta_{ij} \; .$$

Problem 3:

A two dimensional isotropic harmonic oscillator, like our Foucault pendulum in the Lars Onsager Building (Realfagbygget), has the Hamiltonian

$$H = \frac{1}{2m}\,(p_1^{\,2} + p_2^{\,2}) + \frac{1}{2}\,m\omega^2 (x_1^{\,2} + x_2^{\,2}) \; ,$$

where m is the mass and ω the angular frequency. The coordinates x_1, x_2 and canonical momenta p_1, p_2 are Hermitean linear operators satisfying the canonical commutation relations

$$[x_j, p_k] = \mathrm{i}\hbar\delta_{jk} \; , \qquad [x_j, x_k] = 0 \; , \qquad [p_j, p_k] = 0 \; .$$

We may choose units for mass, time and length such that $m = 1$, $\omega = 1$, and $\hbar = 1$. Then

$$H = \frac{1}{2}\,(p_1^{\,2} + p_2^{\,2} + x_1^{\,2} + x_2^{\,2}) = a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \; ,$$

with

$$a_k = \frac{1}{\sqrt{2}}\,(x_k + \mathrm{i}p_k) \; , \qquad a_k^\dagger = \frac{1}{\sqrt{2}}\,(x_k - \mathrm{i}p_k) \; , \qquad k = 1, 2 \; . \tag{1}$$

The canonical commutation relations may be written as

$$[a_j, a_k^\dagger] = \delta_{jk} \; , \qquad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0 \; .$$

The ground state $|0\rangle$ is given by the equations

$$a_1 |0\rangle = a_2 |0\rangle = 0 \ .$$

The states

$$|r,s\rangle = \frac{1}{\sqrt{r!s!}} (a_1^\dagger)^r (a_2^\dagger)^s |0\rangle \qquad \text{for} \qquad r,s = 0,1,2,\ldots$$

are orthonormal eigenstates of H, with energy eigenvalues $r+s+1$ in our "natural units".

We see that the energy levels are degenerate, and we would like to explain this degeneracy as the result of some symmetry.

a) An obvious symmetry group is the two dimensional rotation group SO(2), which is generated by the angular momentum operator $L = x_1p_2 - x_2p_1$.

What are the irreducible representations of SO(2)?

Can the rotational symmetry explain the degeneracy of the energy spectrum?

b) Show that

$$L = \mathrm{i}(a_2^\dagger a_1 - a_1^\dagger a_2) \ .$$

Use this expression for L to show that it is Hermitean, and that it commutes with the Hamiltonian $H = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$.

c) We may replace the operators $a_1, a_2, a_1^\dagger, a_2^\dagger$ by

$$a_\pm = \frac{1}{\sqrt{2}} (a_1 \mp \mathrm{i}a_2) \ , \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}} (a_1^\dagger \pm \mathrm{i}a_2^\dagger) \ ,$$

Show that this transformation from one set of operators to another preserves the commutation relations:

$$[a_+ \, , \, a_+^\dagger] = [a_- \, , \, a_-^\dagger] = 1 \ , \qquad \text{all other commutators vanish.} \qquad .$$

Show also that

$$H = a_+^\dagger a_+ + a_-^\dagger a_- + 1 = N_+ + N_- + 1 = N + 1 \ ,$$

and that

$$L = a_+^\dagger a_+ - a_-^\dagger a_- = N_+ - N_- \ ,$$

when we introduce the number operators $N_+ = a_+^\dagger a_+$, $N_- = a_-^\dagger a_-$, and $N = N_+ + N_-$.

d) The pendulum may perform linear oscillations (in a plane) or circular oscillations, or anything in between.

What types of oscillations are excited by the creation operators $a_1^\dagger$, $a_2^\dagger$, $a_+^\dagger$, and $a_-^\dagger$?

e) Introduce the Hermitean operators

$$K_1 = \frac{1}{2} (a_-^\dagger a_+ + a_+^\dagger a_-) \ , \quad K_2 = \frac{\mathrm{i}}{2} (a_-^\dagger a_+ - a_+^\dagger a_-) \ , \quad K_3 = \frac{L}{2} = \frac{1}{2} (a_+^\dagger a_+ - a_-^\dagger a_-) \ .$$

Show that they commute with the Hamiltonian and satisfy the su(2) commutation relations

$$[K_1, K_2] = \mathrm{i}K_3 \ , \qquad [K_2, K_3] = \mathrm{i}K_1 \ , \qquad [K_3, K_1] = \mathrm{i}K_2 \ .$$

What does this tell us about the possible eigenvalues of the angular momentum operator $L = x_1p_2 - x_2p_1$?

f) We have just shown that the two dimensional isotropic harmonic oscillator has an SU(2) symmetry group. We will now show that this explains the degeneracy of the energy spectrum, because the state vectors belonging to one energy level span an irreducible representation of the symmetry group.

An irreducible representation of the Lie group SU(2), or equivalently, of the Lie algebra su(2), is characterized by the eigenvalue of the Casimir operator

$$\vec{K}^{\,2} = K_1^{\,2} + K_2^{\,2} + K_3^{\,2}\ .$$

Find an expression for $\vec{K}^{\,2}$ in terms of the number operator

$$N = N_+ + N_- = a_+^\dagger a_+ + a_-^\dagger a_-\ .$$

From the theory of angular momentum we know that the eigenvalue of $\vec{K}^{\,2}$ in an irreducible representation is $k(k+1)$ with $k = 0, 1/2, 1, 3/2, \ldots$.

What is the dimension of the irreducible representation for a given value of k?

Explain briefly how this accounts for the degeneracy of the energy levels.

g) The purpose of our Foucault pendulum is to show that the Lars Onsager Building is a rotating coordinate system. Assume, for simplicity, that the angular velocity of the Building is a vertical vector $\vec{\Omega}$ (this would be true on the North Pole). When we try to use Newton's laws of motion in a rotating coordinate system, like we do in an inertial system, we have to include what is called an inertial force, or a fictitious force, which is not a proper Newtonian force because it violates Newton's third law. This force is the sum of a centrifugal force, which is the negative gradient of the centrifugal potential

$$V_c = -\frac{1}{2}\, m\Omega^2 (x_1^{\,2} + x_2^{\,2})\ ,$$

and a Coriolis force $2m\,\vec{v} \times \vec{\Omega}$, depending on the velocity $\vec{v}$, similar to the force on an electric charge in a magnetic field and hence derivable from a vector potential. The Hamiltonian of the Foucault pendulum in the rotating coordinate system is

$$H' = \frac{1}{2m}\left((p_1 + m\Omega x_2)^2 + (p_2 - m\Omega x_1)^2\right) + \frac{1}{2}\, m(\omega^2 - \Omega^2)(x_1^{\,2} + x_2^{\,2})\ .$$

Show that

$$H' = H - \Omega L = (1 - \Omega)\, a_+^\dagger a_+ + (1 + \Omega)\, a_-^\dagger a_- + 1.$$

Describe the eigenvalue spectrum of H'.

Which of the su(2) generators commute with H'?

In other words, what is the symmetry group of H'?

h) Why does the oscillation plane of the Foucault pendulum rotate?