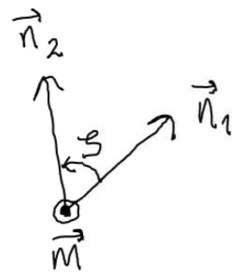
FY3105/8104 Application of symmetry groups in physics
Solution to exam, December 2017

Problem 1

(a) Each conjugacy class consists of all rotations by a given rotation angle, regardless of the rotation axis. The conjugacy classes are therefore labeled by the rotation angle η.

Justification: Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be two arbitrary rotation axes. Then there is a rotation axis $\boldsymbol{m}$ that is perpendicular to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ such that a rotation by angle ζ around $\boldsymbol{m}$ takes $\boldsymbol{n}_1$ to $\boldsymbol{n}_2$ (as illustrated in the figure, where the axis $\boldsymbol{m}$ points out of the paper). It can then be seen that

$$R(\boldsymbol{n}_2,\eta) = R(\boldsymbol{m},\zeta)R(\boldsymbol{n}_1,\eta)R^{-1}(\boldsymbol{m},\zeta). \tag{1}$$



This relation shows that the two rotations $R(\boldsymbol{n}_1,\eta)$ and $R(\boldsymbol{n}_2,\eta)$, which have the same rotation angle η but different rotation axes, are conjugate to each other (with $R(\boldsymbol{m},\zeta)$ as the conjugating element) and are therefore in the same conjugacy class.

(b) We have

$$D^{(j)}_{m'm}(R) = \langle j,m'|U(R)|j,m\rangle. \tag{2}$$

(This follows from

$$\langle j'',m''|U(R)|j,m\rangle = \sum_{m'}\langle j'',m''|j,m'\rangle D^{(j)}_{m'm}(R) = \sum_{m'}\delta_{j'',j}\delta_{m'',m'}D^{(j)}_{m'm}(R) = \delta_{j'',j}D^{(j)}_{m''m}(R), \tag{3}$$

where we used the orthogonality of basis states, $\langle j'',m''|j,m'\rangle = \delta_{j'',j}\delta_{m'',m'}$.) For a rotation around the z axis ($\boldsymbol{n}=\hat{\boldsymbol{z}}$), $\boldsymbol{J}\cdot\boldsymbol{n} = J_z$. Since $J_z|j,m\rangle = m\hbar|j,m\rangle$, we get $\exp(-iJ_z\eta/\hbar)|j,m\rangle = \sum_{r=0}^{\infty}\frac{(-i\eta/\hbar)^r}{r!}J_z^r|j,m\rangle = \sum_{r=0}^{\infty}\frac{(-i\eta/\hbar)^r}{r!}(m\hbar)^r|j,m\rangle = \exp(-im\eta)|j,m\rangle$. Thus

$$D^{(j)}_{m'm}(R) = D^{(j)}_{m'm}(\hat{\boldsymbol{z}},\eta) = \langle j,m'|\exp(-iJ_z\eta/\hbar)|j,m\rangle = \exp(-im\eta)\langle j,m'|j,m\rangle = e^{-im\eta}\delta_{m'm}. \tag{4}$$

(c) Since $\chi(R)$ is a function only of the conjugacy class to which R belongs, it follows from (a) that $\chi(R)$ will only depend on the rotation angle η, not the rotation axis $\boldsymbol{n}$. To evaluate $\chi(R)$ we may therefore pick $\boldsymbol{n}$ at will. In light of the calculation done in (b), it is clearly convenient to pick $\boldsymbol{n}=\hat{\boldsymbol{z}}$. Thus

$$\begin{aligned}\chi^{(j)}(R) &= \mathrm{Tr}\ D^{(j)}(R(\hat{\boldsymbol{z}},\eta)) = \sum_{m=-j}^{j}e^{-im\eta} = \sum_{r=0}^{2j}(e^{-i\eta})^{-j+r} = e^{ij\eta}\sum_{r=0}^{2j}(e^{-i\eta})^r \\ &= e^{ij\eta}\frac{1-(e^{-i\eta})^{2j+1}}{1-e^{-i\eta}} = \frac{e^{ij\eta}-e^{-ij\eta}e^{-i\eta}}{1-e^{-i\eta}} = \frac{e^{-i\eta/2}(e^{i(j+1/2)\eta}-e^{-i(j+1/2)\eta})}{e^{-i\eta/2}(e^{i\eta/2}-e^{-i\eta/2})} = \frac{\sin((j+1/2)\eta)}{\sin(\eta/2)}.\end{aligned} \tag{5}$$

Here we used $\sum_{r=0}^{n}x^r = (1-x^{n+1})/(1-x)$ (sum of a geometric series) and $e^{iy}-e^{-iy} = 2i\sin y$.

(d) Taking $\boldsymbol{n}=\hat{\boldsymbol{z}}$ and considering an infinitesimal rotation angle $d\eta$, the left-hand side is

$$U(R)T^{(k)}_q U^\dagger(R) = (I - iJ_z d\eta/\hbar + \ldots)T^{(k)}_q(I + iJ_z d\eta/\hbar + \ldots) = T^{(k)}_q - i(d\eta/\hbar)[J_z,T^{(k)}_q] + O((d\eta)^2), \tag{6}$$

and the right-hand side is

$$\sum_{q'}T^{(k)}_{q'}D^{(k)}_{q'q}(R) = \sum_{q'}T^{(k)}_{q'}e^{-iqd\eta}\delta_{q'q} = T^{(k)}_q e^{-iqd\eta} = T^{(k)}_q(1 - iqd\eta + \ldots) = T^{(k)}_q - iqd\eta T^{(k)}_q + O((d\eta)^2), \tag{7}$$

where we used (4). For each power of $d\eta$, the coefficients on the two sides must be the same. We see that this holds trivially for the zeroth order terms, for which the coefficient is $T^{(k)}_q$ on both sides. Equating the coefficients of the first-order terms gives $(-i/\hbar)[J_z, T_q^{(k)}] = -iqT_q^{(k)}$, i.e. $[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$. QED.

(e) A rank-0 spherical tensor operator has $k = 0$, so q can only be 0. Furthermore, since $\boldsymbol{r}$ and $\boldsymbol{p}$ are rank-1 spherical tensor operators, we have $k_1 = k_2 = 1$, so q_1 and q_2 can take values $-1, 0, 1$. Taking $X \to \boldsymbol{r}$ and $Z \to \boldsymbol{p}$ thus gives

$$T_0^{(0)} = \sum_{q_1,q_2} \langle 11; q_1 q_2 | 11; 00 \rangle r_{q_1} p_{q_2} = \sum_{q_1=-1}^{1} \langle 11; q_1, -q_1 | 11; 00 \rangle r_{q_1} p_{-q_1}, \tag{8}$$

where in the last step we used that the Clebsch-Gordan coefficients vanish unless $q_1 + q_2 = q$, i.e. $q_2 = -q_1$ in our case. Using the tables of Clebsch-Gordan coefficients we find

$$T_0^{(0)} = \frac{1}{\sqrt{3}} (r_{-1} p_1 - r_0 p_0 + r_1 p_{-1}). \tag{9}$$

Inserting the given expressions for the spherical components of $\boldsymbol{r}$ and $\boldsymbol{p}$ in terms of the cartesian ones, one finds, after some cancellations,

$$T_0^{(0)} = -\frac{1}{\sqrt{3}} (r_x p_x + r_y p_y + r_z p_z) = -\frac{1}{\sqrt{3}} \boldsymbol{r} \cdot \boldsymbol{p}. \tag{10}$$

Thus $T_0^{(0)}$ is proportional to the dot product (scalar product) $\boldsymbol{r} \cdot \boldsymbol{p}$. This is a very reasonable result, since a dot product is invariant under rotations, i.e. it is a scalar (as implied by "scalar product"), and a rank-0 spherical tensor is a scalar operator.[1,2]

Problem 2

(a) (i) We do the proof of $b_1 c = b_3$ graphically by showing that the effect of the two sides on a triangle are identical (see the figure below).

(ii) There are several ways to find the missing elements in the multiplication table. One could also here use the graphical proof method used in (i). Alternatively, one can rewrite the relevant unknown products in terms of known products. Doing this for the unknown products in the third row gives

$$c^2 b_1 = c(cb_1) = cb_2 = b_3, \tag{11}$$
$$c^2 b_2 = c(cb_2) = cb_3 = b_1, \tag{12}$$
$$c^2 b_3 = c(cb_3) = cb_1 = b_2. \tag{13}$$

The missing elements in the fourth row can then be found by applying the rearrangement theorem to each of the last three columns. This gives $b_1 b_1 = e$, $b_1 b_2 = c^2$, and $b_1 b_3 = c$. Alternatively, it is obvious

[1] It can be seen from the defining transformation law (Eq. (6) in the problem text) that a rank-k spherical tensor operator has $2k + 1$ components that transform among themselves under rotations according to the irrep $D^{(k)}$. For $k = 0$ there is thus only one component, which transforms according to the 1-dimensional irrep $D^{(0)}$, which is the trivial irrep, and so $T_0^{(0)}$ is invariant under rotations, i.e. it is a scalar. Equivalently, this can also be seen from the commutators $[J_z, T_q^{(k)}]$ and $[J_\pm, T_q^{(k)}]$. For $k = 0$ one finds (see the formula set) that all three commutators vanish, and thus $T_0^{(0)}$ commutes with the generator of infinitesimal rotations $\boldsymbol{J} \cdot \boldsymbol{n} = J_x n_x + J_y n_y + J_z n_z$ around any rotation axis $\boldsymbol{n}$, and consequently $T_0^{(0)}$ is invariant under all rotations.

[2] In this example we took $X \to \boldsymbol{r}$ and $Z \to \boldsymbol{p}$. The opposite choice would have given $-\boldsymbol{p} \cdot \boldsymbol{r}/\sqrt{3}$, which is obviously also a perfectly valid rank-0 spherical tensor operator. Thus $\boldsymbol{r} \cdot \boldsymbol{p}$ and $\boldsymbol{p} \cdot \boldsymbol{r}$ can differ only by another scalar, which can be found from the commutation relations $[r_j, p_k] = i\hbar\delta_{jk}$ to be $3i\hbar$. Using the symmetric combination $\boldsymbol{r} \cdot \boldsymbol{p} + \boldsymbol{p} \cdot \boldsymbol{r}$ would give a rank-0 spherical tensor operator that is also hermitian.

that $b_1 b_1 = e$ since b_i^2 is a 2π rotation which thus is equivalent to the identity element. And since the π rotations b_i therefore are their own inverses, we get

$$b_1 b_2 = b_1^{-1} b_2^{-1} = (b_2 b_1)^{-1} = c^{-1} = c^2, \quad (14)$$
$$b_1 b_3 = b_1^{-1} b_3^{-1} = (b_3 b_1)^{-1} = (c^2)^{-1} = c. \quad (15)$$

The missing elements in the third row can now be found from the rearrangement theorem. Alternatively, one can deduce all six missing elements entirely from the rearrangement theorem. For example, this theorem gives that the missing elements in the third row must be b_1, b_2 and b_3. In the fourth column, b_1 and b_2 are excluded since they already appear in it, so b_3 must appear at the intersection of the third row and fourth column. Then e must be the other missing element in the fourth column. Similar reasoning easily gives the missing elements in the fifth and sixth columns.

In conclusion, the complete multiplication table for the group D_3 is shown below.

	e	c	c^2	b_1	b_2	b_3
e	e	c	c^2	b_1	b_2	b_3
c	c	c^2	e	b_2	b_3	b_1
c^2	c^2	e	c	b_3	b_1	b_2
b_1	b_1	b_3	b_2	e	c^2	c
b_2	b_2	b_1	b_3	c	e	c^2
b_3	b_3	b_2	b_1	c^2	c	e

(b) A conjugacy class of a group G consists of a set of elements, such that if a and b are in this set, we have $b = gag^{-1}$ for some $g \in G$.

The identity element e is a conjugacy class by itself, since $geg^{-1} = e$ for any g. Next, let us identify elements conjugate to c. The multiplication table gives

$$ece^{-1} = c, \quad (16)$$
$$ccc^{-1} = ccc^2 = c^4 = c, \quad (17)$$
$$c^2 c (c^2)^{-1} = c^2 cc = c^4 = c, \quad (18)$$
$$b_1 c b_1^{-1} = b_1 c b_1 = b_1 b_2 = c^2, \quad (19)$$
$$b_2 c b_2^{-1} = b_2 c b_2 = b_2 b_3 = c^2, \quad (20)$$
$$b_3 c b_3^{-1} = b_3 c b_3 = b_3 b_1 = c^2, \quad (21)$$

which shows that $\{c, c^2\}$ is a conjugacy class. Furthermore,

$$c b_1 c^{-1} = c b_1 c^2 = c b_2 = b_3, \quad (22)$$
$$c^2 b_1 (c^2)^{-1} = c^2 b_1 c = c^2 b_3 = b_2, \quad (23)$$

from which we conclude that $\{b_1, b_2, b_3\}$ is a conjugacy class. As each element of the group has now been placed in a conjugacy class, we conclude that D_3 has 3 conjugacy classes: $\{e\}$, $\{c, c^2\}$, and $\{b_1, b_2, b_3\}$.[3]

(c) According to Lagrange's theorem, the order of a subgroup must be a divisor of the order $|G|$ of the group G itself. Any group G has the subgroups G and $\{e\}$, of order $|G|$ and 1, respectively. Since D_3 has order 6, any other subgroups must have order 2 or 3. As D_3 is a finite group, the distinct powers of any element form a subgroup. For the π rotations b_i we have $b_i^2 = e$, so $\{b_i, e\}$ is a subgroup of order 2 for $i = 1, 2, 3$. Also, the distinct powers of c, i.e. $\{c, c^2, c^3 = e\}$ is a subgroup of order 3; the distinct powers of c^2 give the same subgroup. In conclusion, D_3 has six subgroups: D_3, $\{e, c, c^2\}$, $\{e, b_1\}$, $\{e, b_2\}$, $\{e, b_3\}$, $\{e\}$.

[3]That c and c^2 cannot be in the same conjugacy class(es) as b_1, b_2, and b_3 could have been concluded from the outset from the fact that D_3 is a subgroup of SO(3), for which rotations by different angles are in different conjugacy classes (unless the angles add to a multiple of 2π).

(d) As the number of irreps equals the number of conjugacy classes, it follows from (b) that D_3 has 3 irreps. We denote these by $\Gamma^{(\alpha)}$ ($\alpha = 1, 2, 3$). These label the rows of the character table. The columns are labeled by the conjugacy classes, since the characters are the same for all elements of a conjugacy class.

The irrep dimensions d_α satisfy $\sum_\alpha d_\alpha^2 = |G|$ (see formula set), which for D_3 becomes

$$d_1^2 + d_2^2 + d_3^2 = 6. \tag{24}$$

It follows that two dimensions equal 1 and one dimension equals 2. We pick $d_1 = d_2 = 1$ and $d_3 = 2$.

Any representation is a homomorphism and must therefore satisfy

$$\Gamma(g_1 g_2) = \Gamma(g_1)\Gamma(g_2). \tag{25}$$

Taking $g_2 = e$ gives $\Gamma(g_1 e) = \Gamma(g_1) = \Gamma(g_1)\Gamma(e)$. Multiplying with $(\Gamma(g_1))^{-1}$ from the left gives $\Gamma(e) = I$, i.e. the identity element is always represented by the unit matrix. Thus $\chi^{(\alpha)}(e) = d_\alpha = 1$, 1, 2 for $\alpha = 1, 2, 3$. This gives the left column of the character table.

For any group, $\Gamma(g) = 1$ for all g is a possible solution of (25). As this is a 1-dimensional representation, it is irreducible. It is called the trivial irrep; we take this to be $\Gamma^{(1)}$. Thus the characters $\chi^{(1)}(g) = 1$ too. This completes the top row of the character table.

For 1-dimensional irreps, $\chi(g) = \Gamma(g)$, and so the characters themselves must satisfy the homomorphism property (25), i.e. $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$. Taking $g_1 = g_2 = b_1$, this gives $\chi(b_1^2) = (\chi(b_1))^2$. Using $b_1^2 = e$ and $\chi(e) = 1$ then gives $\chi(b_1) = \pm 1$. Next, taking $g_1 = b_1$ and $g_2 = c$ gives $\chi(b_1 c) = \chi(b_1)\chi(c)$. Using $b_1 c = b_3$, and $\chi(b_1) = \chi(b_3) = \pm 1$, this nonzero character can then be cancelled to give $\chi(c) = 1$. This holds for both the 1d irreps. Since $\Gamma^{(1)}$ has $\chi(b_I) = 1$, $\Gamma^{(2)}$ must then have the remaining alternative $\chi(b_I) = -1$ in order to be distinct from $\Gamma^{(1)}$. This completes the second row in the character table.

To complete the third row in the character table we use the orthogonality relation for columns/classes (see formula set; I denote the classes by k and k' as the symbol c is already used for a group element),

$$\sum_{\alpha=1}^{3} e_k \chi_k^{(\alpha)*} \chi_{k'}^{(\alpha)} = |G|\delta_{kk'}. \tag{26}$$

Taking $k = \{e\}$ and $k' = \{c, c^2\}$ gives $1 \cdot (1 \cdot 1 + 1 \cdot 1 + 2 \cdot \chi_{k'}^{(3)}) = 0$, so $\chi_{k'}^{(3)} = -1$. Instead taking $k' = \{b_1, b_2, b_3\}$ gives $1 \cdot (1 \cdot 1 + 1 \cdot (-1) + 2 \cdot \chi_{k'}^{(3)}) = 0$, so $\chi_{k'}^{(3)} = 0$. This completes the character table:

	$\{e\}$	$\{c, c^2\}$	$\{b_1, b_2, b_3\}$
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	1	-1
$\Gamma^{(3)}$	2	-1	0

(e) The kernel K of a group homomorphism $f : A \to B$ is the set of elements in the source group A that are mapped to the identity element in the target group B: $K = \{a \in A \mid f(A) = e_B\}$. The kernel K is a subgroup of A that is also normal. A subgroup is normal if it consists of complete conjugacy classes.[4]

For (matrix) representations, the target group is a matrix group, whose identity element is the unit matrix. So for each irrep we wish to identify the elements that are mapped to the unit matrix.

For 1-dimensional irreps, the unit matrix is the number 1, and $\chi(g) = \Gamma(g)$, so we can read $\Gamma(g)$ directly from the character table. In this way we can determine the kernels for the 1d irreps $\Gamma^{(1)}$ and $\Gamma^{(2)}$. For $\Gamma^{(1)}$ we see that all elements are mapped to 1, i.e. the kernel $K_1 = D_3$, the whole group, which is obviously a normal subgroup since it consists of all the conjugacy classes. For $\Gamma^{(2)}$ we see that the kernel

[4]Equivalently, a normal subgroup is characterized by left and right cosets being identical. But the equivalent criterion based on complete conjugacy classes is much more practical for checking "normality" in our case.

is $K_2 = \{e, c, c^2\}$. We identified this as a subgroup in 2(c), and it consists of the two conjugacy classes $\{e\}$ and $\{c, c^2\}$, so it is also normal.

For the 2-dimensional irrep $\Gamma^{(3)}$, we have $\chi^{(3)}(e) = 2$. No other elements have character 2 for this irrep, so the kernel is $K_3 = \{e\}$. This is a subgroup consisting of the single conjugacy class $\{e\}$, so it is also normal.

(f) As mentioned before, the identity element e is always represented by the unit matrix. Since Γ is a 3-dimensional representation, $\Gamma(e)$ is therefore the 3×3 unit matrix.

There are several methods that can be used to determine the matrices for the remaining three elements. One method is to make use of the multiplication table and the homomorphism property (25) to express each matrix as a product of known matrices. For example,

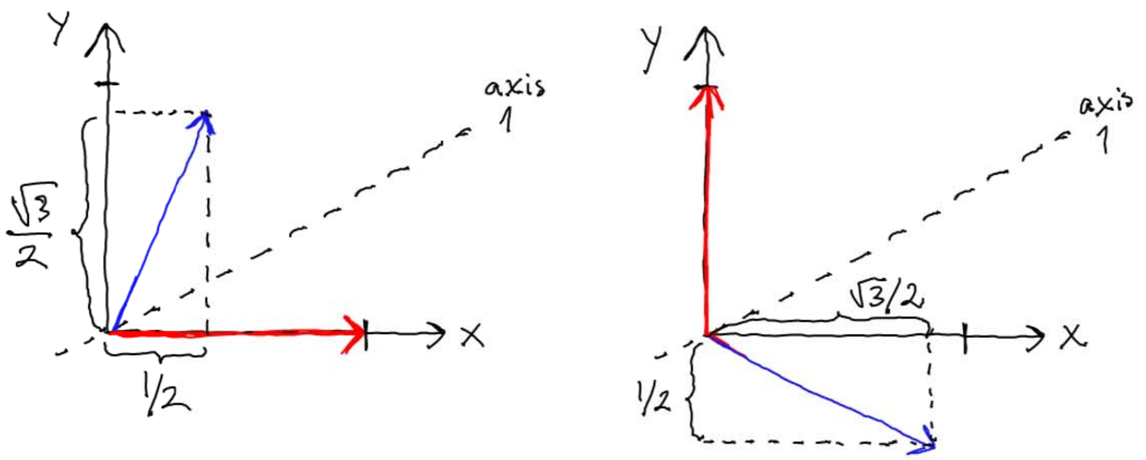
$$c^2 = cc \quad \Rightarrow \quad \Gamma(c^2) = \Gamma(c)\Gamma(c), \tag{27}$$
$$b_1 = cb_3 \quad \Rightarrow \quad \Gamma(b_1) = \Gamma(c)\Gamma(b_3), \tag{28}$$
$$b_2 = b_3 c \quad \Rightarrow \quad \Gamma(b_2) = \Gamma(b_3)\Gamma(c). \tag{29}$$

In this way the unknown matrices can be found simply by matrix multiplication.

Another method is based on the transformation rule for basis vectors (see formula sheet), $a\phi_i = \sum_j \phi_j \Gamma_{ji}(a)$, which gives[5] $\Gamma_{ji}(a) = (\phi_j, a\phi_i)$. Thus we need to know how the group element a transforms the orthonormal basis vectors, which are $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ in our problem. I will illustrate this method by finding $\Gamma(b_1)$. We have

$$b_1 \hat{\boldsymbol{x}} = \frac{1}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}, \tag{30}$$
$$b_1 \hat{\boldsymbol{y}} = \frac{\sqrt{3}}{2}\hat{\boldsymbol{x}} - \frac{1}{2}\hat{\boldsymbol{y}}, \tag{31}$$
$$b_1 \hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}. \tag{32}$$

Here, (30) and (31) follow from the figure below (the red arrow is the basis vector of interest ($\hat{\boldsymbol{x}}$ in the



left panel, $\hat{\boldsymbol{y}}$ in the right panel) and the blue arrow is the transformed basis vector ($b_1\hat{\boldsymbol{x}}$ in the left panel, $b_1\hat{\boldsymbol{y}}$ in the right panel)). Furthermore, (32) follows since b_1 is a π-rotation around a rotation axis in the

[5]Derivation: $(\phi_k, a\phi_i) = \sum_j (\phi_k, \phi_j)\Gamma_{ji}(a) = \sum_j \delta_{kj}\Gamma_{ji}(a) = \Gamma_{ki}(a)$.

xy plane. This gives

$$
\begin{aligned}
\Gamma_{j1}(b_1) &= (\phi_j, b_1\phi_1) = (\phi_j, b_1\hat{\boldsymbol{x}}) = (\phi_j, \frac{1}{2}\hat{\boldsymbol{x}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{y}}) = \frac{1}{2}(\phi_j, \hat{\boldsymbol{x}}) + \frac{\sqrt{3}}{2}(\phi_j, \hat{\boldsymbol{y}}) \\
&\Rightarrow \Gamma_{11}(b_1) = \frac{1}{2}, \quad \Gamma_{21}(b_1) = \frac{\sqrt{3}}{2}, \quad \Gamma_{31}(b_1) = 0, \qquad (33) \\
\Gamma_{j2}(b_1) &= (\phi_j, b_1\phi_2) = (\phi_j, b_1\hat{\boldsymbol{y}}) = (\phi_j, \frac{\sqrt{3}}{2}\hat{\boldsymbol{x}} - \frac{1}{2}\hat{\boldsymbol{y}}) = \frac{\sqrt{3}}{2}(\phi_j, \hat{\boldsymbol{x}}) - \frac{1}{2}(\phi_j, \hat{\boldsymbol{y}}) \\
&\Rightarrow \Gamma_{12}(b_1) = \frac{\sqrt{3}}{2}, \quad \Gamma_{22}(b_1) = -\frac{1}{2}, \quad \Gamma_{32}(b_1) = 0, \qquad (34) \\
\Gamma_{j3}(b_1) &= (\phi_j, b_1\phi_3) = (\phi_j, b_1\hat{\boldsymbol{z}}) = (\phi_j, -\hat{\boldsymbol{z}}) = -(\phi_j, \hat{\boldsymbol{z}}) \\
&\Rightarrow \Gamma_{13}(b_1) = 0, \quad \Gamma_{23}(b_1) = 0, \quad \Gamma_{33}(b_1) = -1. \qquad (35)
\end{aligned}
$$

The matrix $\Gamma(b_1)$ can then be written down from its 9 matrix elements worked out here.

Other methods may also be possible, including working out the matrices as special cases of general rotation matrices in SO(3). Regardless of the method, one arrives at the following matrices:

$$
\begin{aligned}
\Gamma(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma(c^2) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\Gamma(b_1) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Gamma(b_2) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad (36)
\end{aligned}
$$

All matrices of this representation have determinant 1. This can be understood from the fact that the matrices determine the transformation properties of vectors in 3 dimensions under the rotations of D_3 and thus constitute a subgroup of the rotation matrices of (the defining, faithful representation of) SO(3), which have determinant 1.[6]

(g) Γ is a 3-dimensional representation of D_3. From the entries in the leftmost column of the character table, which are the dimensions of the irreps of D_3, one sees that D_3 has only 1- and 2-dimensional irreps. Thus Γ must be reducible.

The decomposition of Γ into irreps of D_3 can be written $\Gamma = \oplus_{\alpha=1}^{3} a_\alpha \Gamma^{(\alpha)}$, where

$$
a_\alpha = \frac{1}{|G|} \sum_{A \in G} \chi^{(\alpha)*}(A)\chi(A). \qquad (37)
$$

The characters of the representation Γ are easily seen to be $\chi(e) = 1 + 1 + 1 = 3$, $\chi(c) = \chi(c^2) = -1/2 - 1/2 + 1 = 0$, and $\chi(b_1) = \chi(b_2) = \chi(b_3) = -1$. This gives

$$
\begin{aligned}
a_1 &= \frac{1}{6}\left[1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3 \cdot 1 \cdot (-1)\right] = 0, \qquad (38) \\
a_2 &= \frac{1}{6}\left[1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3 \cdot (-1) \cdot (-1)\right] = 1, \qquad (39) \\
a_3 &= \frac{1}{6}\left[1 \cdot 2 \cdot 3 + 2 \cdot (-1) \cdot 0 + 3 \cdot 0 \cdot (-1)\right] = 1, \qquad (40) \\
&\Rightarrow \Gamma = \Gamma^{(2)} \oplus \Gamma^{(3)}. \qquad (41)
\end{aligned}
$$

Checking the dimensions of (41) to be on the safe side, we see that the rhs has total dimension

[6] In order to preserve dot products of vectors, it can be shown that the determinant must be ± 1, and since rotation matrices are continuously connected to the unit matrix representing zero rotation, which has determinant 1, general rotation matrices have determinant 1 too.

$d_2 + d_3 = 1 + 2$, which agrees with the dimension $d_\Gamma = 3$ of the lhs.[7]

(h) Since $D^{(1)}$ is a single-valued representation of SO(3), its matrices satisfy the homomorphism property (25), which in this context reads

$$D^{(1)}(R_1 R_2) = D^{(1)}(R_1) D^{(1)}(R_2), \tag{42}$$

where R_1 and R_2 are arbitrary rotations of SO(3). As all elements of the group D_3 are rotations around the origin, D_3 is a subgroup of SO(3), and so (42) clearly holds also if R_1 and R_2 are restricted to the elements of D_3. But then (42) becomes simply the defining property of a representation of D_3. QED.

Two representations are equivalent if their characters are identical. (This condition also contains within itself the necessary but not sufficient condition that the dimensions of the two representations must be identical, since the dimension equals the character of the unit matrix representing the identity element.) We have already calculated the characters χ of the representation Γ in 2(g), so it just remains to calculate the characters $\chi^{(1)}(\eta)$ of the representation $D^{(1)}$, for the rotation angles η of the elements of D_3, and check that they agree. For this we use the already encountered Eq. (5) in the problem text (also in the formula set), which specialized to $j = 1$ becomes $\chi^{(1)}(\eta) = \sin(3\eta/2)/\sin(\eta/2)$. This gives

$$\begin{aligned}
\chi^{(1)}(0) &= \lim_{\eta\to 0} \frac{\sin(3\eta/2)}{\sin(\eta/2)} = \lim_{\eta\to 0} \frac{3\eta/2}{\eta/2} = 3 = \chi(e), && (43)\\
\chi^{(1)}(2\pi/3) &= \frac{\sin(\pi)}{\sin(\pi/3)} = 0 = \chi(c), && (44)\\
\chi^{(1)}(\pi) &= \frac{\sin(3\pi/2)}{\sin(\pi/2)} = \frac{-1}{1} = -1 = \chi(b_i). && (45)
\end{aligned}$$

Thus we see that the relevant characters of $D^{(1)}$ indeed all agree with those of Γ.[8]

(i) We need to decompose $D^{(2)}$ into the irreps of D_3: $D^{(2)} = \oplus_{\alpha=1}^{3} a_\alpha \Gamma^{(\alpha)}$. Thus we need to calculate $\chi^{(2)}(\eta) = \sin(5\eta/2)/\sin(\eta/2)$ for the same values of η as in 2(h). We find

$$\begin{aligned}
\chi^{(2)}(0) &= \lim_{\eta\to 0} \frac{\sin(5\eta/2)}{\sin(\eta/2)} = \lim_{\eta\to 0} \frac{5\eta/2}{\eta/2} = 5, && (46)\\
\chi^{(2)}(2\pi/3) &= \frac{\sin(5\pi/3)}{\sin(\pi/3)} = \frac{-\sqrt{3}/2}{\sqrt{3}/2} = -1, && (47)\\
\chi^{(2)}(\pi) &= \frac{\sin(5\pi/2)}{\sin(\pi/2)} = \frac{1}{1} = 1. && (48)
\end{aligned}$$

This gives

$$\begin{aligned}
a_1 &= \frac{1}{6}\left[1\cdot 1\cdot 5 + 2\cdot 1\cdot(-1) + 3\cdot 1\cdot 1\right] = 1, && (49)\\
a_2 &= \frac{1}{6}\left[1\cdot 1\cdot 5 + 2\cdot 1\cdot(-1) + 3\cdot(-1)\cdot 1\right] = 0, && (50)\\
a_3 &= \frac{1}{6}\left[1\cdot 2\cdot 5 + 2\cdot(-1)\cdot(-1) + 3\cdot 0\cdot 1\right] = 2, && (51)\\
&\Rightarrow D^{(2)} = \Gamma^{(1)} \oplus 2\Gamma^{(3)} = \Gamma^{(1)} \oplus \Gamma^{(3)} \oplus \Gamma^{(3)}. && (52)
\end{aligned}$$

(Let us again check the dimensions. The total dimension on the rhs is $d_1 + 2d_3 = 1 + 2\cdot 2 = 5$, which is also the dimension of $D^{(2)}$.) We conclude that the five-fold degenerate $D^{(2)}$ level splits into three levels; one level belongs to the irrep $\Gamma^{(1)}$ and is therefore nondegenerate, while the other two levels belong to the irrep $\Gamma^{(3)}$ and are therefore each two-fold degenerate.

[7] The result (41) also makes sense by looking at the matrices in the representation Γ, which have an explicitly block-diagonal structure with a 2×2 "xy block" and a 1×1 "z block" , and it is evident that the z coordinate transforms according to the irrep $\Gamma^{(2)}$ (invariant under e, c and c^2, while changing sign under b_1, b_2, and b_3), and so the xy coordinates therefore transform according to the 2-dimensional irrep $\Gamma^{(3)}$.

[8] Since c^2 rotates by an angle $4\pi/3$, you may also want to check that $\chi^{(1)}(4\pi/3) = 0$, which indeed is the case. Note that $4\pi/3 = 2\pi - 2\pi/3$. One can show that for j integer, i.e. for the single-valued irreps of SO(3), $\chi^{(j)}(2\pi - \eta) = \chi^{(j)}(\eta)$.