Eksamination FY8104 Symmetry in physics
Tuesday December 13, 2011
Solutions

1) The four-group is commutative (Abelian), so each element is its own conjugation class, and the irreducible representations are one dimensional. There are 4 conjugation classes and the same number of irreducible representations.

A one dimensional character χ is a representation. Since the elements a, b, c have order 2, $a^2 = b^2 = c^2 = e$, we must have for $g = a, b, c$ that

$$(\chi(g))^2 = \chi(g^2) = \chi(e) = 1 \ ,$$

hence $\chi(g) = \pm 1$. We must also have that

$$\chi(a)\,\chi(b) = \chi(ab) = \chi(c) \ .$$

This gives the following character table:

	e	a	b	c
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

We may check the orthogonality relations for the rows and the columns.

A faithful representation is one where all the different group elements are represented differently. There is no faithful irreducible representation, we see from the character table that at least two group elements are represented by the number 1 in any irreducible representation.

A reducible representation of this group is faithful if it contains at least two of the three irreducible representations with characters χ_2, χ_3, χ_4. For example a two dimensional matrix representation $\mathbf{D}$ containing χ_2 and χ_3,

$$\mathbf{D}(g) = \begin{pmatrix} \chi_2(g) & 0 \\ 0 & \chi_3(g) \end{pmatrix} \qquad \text{for} \qquad g = e, a, b, c \ ,$$

which means that

$$\mathbf{D}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \mathbf{D}(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \mathbf{D}(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \mathbf{D}(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

2a) The order of a group element must be a factor of 16, either 1, 2, 4, 8, or 16. There are

1 element of order 1: $\mathbf{I}$;

7 elements of order 2: $-\mathbf{I}$, $\pm\boldsymbol{\sigma}_1$, $\pm\boldsymbol{\sigma}_2$, $\pm\boldsymbol{\sigma}_3$;

8 elements of order 4: $\pm\mathrm{i}\mathbf{I}$, $\pm\mathrm{i}\boldsymbol{\sigma}_1$, $\pm\mathrm{i}\boldsymbol{\sigma}_2$, $\pm\mathrm{i}\boldsymbol{\sigma}_3$.

The elements $\pm\mathbf{I}$ and $\pm\mathrm{i}\mathbf{I}$ commute with all group elements, they are their own conjugation classes (one group element in each class).

The elements $\pm\boldsymbol{\sigma}_1$ are conjugate, we have for example that $\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2)^{-1} = -\boldsymbol{\sigma}_1$.

Thus there are 6 conjugation classes with two elements in each:
$\{\pm\boldsymbol{\sigma}_1\}$, $\{\pm\boldsymbol{\sigma}_2\}$, $\{\pm\boldsymbol{\sigma}_3\}$, $\{\pm\mathrm{i}\boldsymbol{\sigma}_1\}$, $\{\pm\mathrm{i}\boldsymbol{\sigma}_2\}$, $\{\pm\mathrm{i}\boldsymbol{\sigma}_3\}$.

2b) The possible orders of subgroups are 1, 2, 4, 8, and 16. The subgroups of order 1 and 16 are the trivial ones: the unit element $\mathbf{I}$ alone, and the whole group.

There are 7 subgroups of order 2, one for each of the 7 elements of order 2. They are: $\{\pm\mathbf{I}\}$, $\{\mathbf{I},\boldsymbol{\sigma}_1\}$, $\{\mathbf{I},-\boldsymbol{\sigma}_1\}$, $\{\mathbf{I},\boldsymbol{\sigma}_2\}$, $\{\mathbf{I},-\boldsymbol{\sigma}_2\}$, $\{\mathbf{I},\boldsymbol{\sigma}_3\}$, $\{\mathbf{I},-\boldsymbol{\sigma}_3\}$.

The subgroup $\{\mathbf{I},-\mathbf{I}\}$ is normal, it contains complete conjugation classes. The subgroup $\{\mathbf{I},\boldsymbol{\sigma}_1\}$, for example, is not normal, it contains only half of the conjugation class $\{\pm\boldsymbol{\sigma}_1\}$.

There are many subgroups of order 4. One is $\{\pm\mathbf{I},\pm\mathrm{i}\mathbf{I}\}$. It is cyclic, generated by either $\mathrm{i}\mathbf{I}$ or $-\mathrm{i}\mathbf{I}$. This is the centre of the group, defined as the set of group elements commuting with all group elements. The centre of a group is always a normal subgroup.

Other cyclic subgroups of order 4 are $\{\pm\mathbf{I},\pm\mathrm{i}\boldsymbol{\sigma}_1\}$, $\{\pm\mathbf{I},\pm\mathrm{i}\boldsymbol{\sigma}_2\}$, and $\{\pm\mathbf{I},\pm\mathrm{i}\boldsymbol{\sigma}_3\}$. They are also normal, since they contain complete conjugation classes.

Non-cyclic subgroups of order 4 are $\{\pm\mathbf{I},\pm\boldsymbol{\sigma}_1\}$, $\{\pm\mathbf{I},\pm\boldsymbol{\sigma}_2\}$, and $\{\pm\mathbf{I},\pm\boldsymbol{\sigma}_3\}$. A non-cyclic group of order 4 is isomorphic to the four-group. Again, these subgroups are normal.

A normal subgroup of order 8 is the quaternion group $\{\pm\mathbf{I},\pm\mathrm{i}\boldsymbol{\sigma}_1,\pm\mathrm{i}\boldsymbol{\sigma}_2,\pm\mathrm{i}\boldsymbol{\sigma}_3\}$.

2c) Altogether, there are 10 conjugation classes and 10 irreducible representations. There have to be 8 one dimensional and 2 two dimensional irreducible representations, this is the only way to make the square sum of the dimensions equal to 16.

In a one dimensional representation the elements of order 2 may have character values ± 1, and the elements of order 4 may have character values $\pm 1, \pm\mathrm{i}$.

If we divide out by the centre of the group, $\{\pm\mathbf{I},\pm\mathrm{i}\mathbf{I}\}$, the factor group is the four-group. The explicit homomorphism is for example like this:

$$\{\pm\mathbf{I},\pm\mathrm{i}\mathbf{I}\}\mapsto e\ , \quad \{\pm\boldsymbol{\sigma}_1,\pm\mathrm{i}\boldsymbol{\sigma}_1\}\mapsto a\ , \quad \{\pm\boldsymbol{\sigma}_2,\pm\mathrm{i}\boldsymbol{\sigma}_2\}\mapsto b\ , \quad \{\pm\boldsymbol{\sigma}_3,\pm\mathrm{i}\boldsymbol{\sigma}_3\}\mapsto c\ .$$

In this way we get the characters $\chi_1,\chi_2,\chi_3,\chi_4$ in the character table below from the character table of the four-group.

If we divide out by the quaternion group, the factor group is the cyclic group of order 2. The character χ_5 in the character table below comes from this factor group.

To finish the one dimensional characters we define $\chi_6(g)=\chi_2(g)\chi_5(g)$, $\chi_7(g)=\chi_3(g)\chi_5(g)$, and $\chi_8(g)=\chi_4(g)\chi_5(g)$ for a group element g.

	$\mathbf{I}$	$-\mathbf{I}$	$\mathrm{i}\mathbf{I}$	$-\mathrm{i}\mathbf{I}$	$\pm\boldsymbol{\sigma}_1$	$\pm\boldsymbol{\sigma}_2$	$\pm\boldsymbol{\sigma}_3$	$\pm\mathrm{i}\boldsymbol{\sigma}_1$	$\pm\mathrm{i}\boldsymbol{\sigma}_2$	$\pm\mathrm{i}\boldsymbol{\sigma}_3$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	1	-1	-1
χ_3	1	1	1	1	-1	1	-1	-1	1	-1
χ_4	1	1	1	1	-1	-1	1	-1	-1	1
χ_5	1	1	-1	-1	-1	-1	-1	1	1	1
χ_6	1	1	-1	-1	-1	1	1	1	-1	-1
χ_7	1	1	-1	-1	1	-1	1	-1	1	-1
χ_8	1	1	-1	-1	1	1	-1	-1	-1	1
χ_9	2	-2	2i	-2i	0	0	0	0	0	0
χ_{10}	2	-2	-2i	2i	0	0	0	0	0	0

A natural guess is that the 2 two dimensional irreducible representations are the defining representation

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and its complex conjugate. This completes the character table, given above.

3a) Let $R(\alpha)$ be a rotation by an angle α. Then $R(\alpha)R(\beta) = R(\beta)R(\alpha) = R(\alpha+\beta)$, and $R(2\pi) = R(0) = I$ (the identity).

The two dimensional rotation group SO(2) is commutative (Abelian), hence its irreducible representations are one dimensional, by Schur's lemma.

An irreducible representation of SO(2) is characterized by an integer quantum number ℓ, such that the rotation $R(\alpha)$ is represented by the complex number $\mathrm{e}^{\mathrm{i}\ell\alpha}$. The physical interpretation is that $\ell\hbar$ is the angular momentum.

A symmetry transformation commutes with the Hamiltonian. Hence, it follows from Schur's lemma that all states belonging to one irreducible representation of a symmetry group must have the same energy. But this does not produce degeneracy in the energy spectrum when all the irreducible representations of the symmetry group are one dimensional.

3b) We have that

$$x_k = \frac{1}{\sqrt{2}}\,(a_k^\dagger + a_k)\;, \qquad p_k = \frac{\mathrm{i}}{\sqrt{2}}\,(a_k^\dagger - a_k)\;, \qquad k = 1, 2\;.$$

This gives that

$$\begin{aligned} L &= x_1p_2 - x_2p_1 = \frac{\mathrm{i}}{2}\,((a_1^\dagger + a_1)(a_2^\dagger - a_2) - (a_2^\dagger + a_2)(a_1^\dagger - a_1)) \\ &= \frac{\mathrm{i}}{2}\,(a_1^\dagger a_2^\dagger - a_1^\dagger a_2 + a_1 a_2^\dagger - a_1 a_2 - a_2^\dagger a_1^\dagger + a_2^\dagger a_1 - a_2 a_1^\dagger + a_2 a_1) \\ &= \mathrm{i}\,(a_2^\dagger a_1 - a_1^\dagger a_2)\;. \end{aligned}$$

L is Hermitean because

$$L^\dagger = -\mathrm{i}\,((a_2^\dagger a_1)^\dagger - (a_1^\dagger a_2)^\dagger) = -\mathrm{i}\,(a_1^\dagger a_2 - a_2^\dagger a_1) = L\;.$$

To see that it commutes with the Hamiltonian we compute

$$\begin{aligned} [-\mathrm{i}L, H] &= [\,a_2^\dagger a_1 - a_1^\dagger a_2\,,\, a_1^\dagger a_1 + a_2^\dagger a_2 + 1\,] \\ &= [a_2^\dagger a_1, a_1^\dagger a_1] + [a_2^\dagger a_1, a_2^\dagger a_2] - [a_1^\dagger a_2, a_1^\dagger a_1] - [a_1^\dagger a_2, a_2^\dagger a_2] \\ &= a_2^\dagger\,[a_1, a_1^\dagger a_1] + [a_2^\dagger, a_2^\dagger a_2]\,a_1 - [a_1^\dagger, a_1^\dagger a_1]\,a_2 - a_1^\dagger\,[a_2, a_2^\dagger a_2] \\ &= a_2^\dagger\,[a_1, a_1^\dagger]\,a_1 + a_2^\dagger\,[a_2^\dagger, a_2]\,a_1 - a_1^\dagger\,[a_1^\dagger, a_1]\,a_2 - a_1^\dagger\,[a_2, a_2^\dagger]\,a_2 \\ &= a_2^\dagger a_1 - a_2^\dagger a_1 + a_1^\dagger a_2 - a_1^\dagger a_2 = 0\;. \end{aligned}$$

The operators a and $a^\dagger$ are known from the quantum theory of the harmonic oscillator as annihilation and creation operators. With this knowledge we may conclude without computing that L commutes with H. The Hamiltonian $H = N_1 + N_2 + 1$ counts the total number of quanta in the two modes 1 and 2, since $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$ are the numbers of energy quanta in the two modes, and since the energy per quantum is the same in the two modes. The operator $a_2^\dagger a_1$ commutes with H because it preserves the total number of quanta: it destroys one quantum in mode 1 and creates one quantum in mode 2. Similarly, $a_1^\dagger a_2$ commutes with H because it destroys one quantum in mode 2 and creates one quantum in mode 1.

3c) We have that

$$\begin{aligned}
[a_+, a_+^\dagger] &= \frac{1}{2}\,[\,a_1 - \mathrm{i}a_2\,,\,a_1^\dagger + \mathrm{i}a_2^\dagger\,] = \frac{1}{2}\,([a_1, a_1^\dagger] + [a_2, a_2^\dagger]) = 1\;,\\
[a_-, a_-^\dagger] &= \frac{1}{2}\,[\,a_1 + \mathrm{i}a_2\,,\,a_1^\dagger - \mathrm{i}a_2^\dagger\,] = \frac{1}{2}\,([a_1, a_1^\dagger] + [a_2, a_2^\dagger]) = 1\;,\\
[a_-, a_+^\dagger] &= \frac{1}{2}\,[\,a_1 + \mathrm{i}a_2\,,\,a_1^\dagger + \mathrm{i}a_2^\dagger\,] = \frac{1}{2}\,([a_1, a_1^\dagger] - [a_2, a_2^\dagger]) = 0\;.
\end{aligned}$$

Since

$$a_1 = \frac{1}{\sqrt{2}}\,(a_+ + a_-)\;, \qquad a_2 = \frac{\mathrm{i}}{\sqrt{2}}\,(a_+ - a_-)\;,$$

we have that

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + 1 = \frac{1}{2}\,((a_+^\dagger + a_-^\dagger)(a_+ + a_-) + (a_+^\dagger - a_-^\dagger)(a_+ - a_-)) + 1 = a_+^\dagger a_+ + a_-^\dagger a_- + 1\;,$$

and that

$$L = \mathrm{i}\,(a_2^\dagger a_1 - a_1^\dagger a_2) = \frac{1}{2}\,((a_+^\dagger - a_-^\dagger)(a_+ + a_-) + (a_+^\dagger + a_-^\dagger)(a_+ - a_-)) = a_+^\dagger a_+ - a_-^\dagger a_-\;.$$

Since $a_+^\dagger a_+$ and $a_-^\dagger a_-$ are number operators having non-negative integer eigenvalues, this proves that the orbital angular momentum $L = x_1 p_2 - x_2 p_1$ has integer eigenvalues, or integer muliples of $\hbar$ if we do not set $\hbar = 1$.

In general, angular momentum may be integer or half integer, but orbital angular momentum has to be integer and can not be half integer. The proof given here is the best (most convincing) proof we have.

3d) $a_1^\dagger$ and $a_2^\dagger$ excite linear oscillations in the x_1 and x_2 direction, respectively.

$a_+^\dagger$ excites circular oscillations with positive angular momentum, that is, in the anticlockwise direction.

$a_-^\dagger$ excites circular oscillations with negative angular momentum, that is, in the clockwise direction.

3e) The operators $a_-^\dagger a_+$ and $a_+^\dagger a_-$ commute with the Hamiltonian $H = a_+^\dagger a_+ + a_-^\dagger a_- + 1$ in the same way as $a_2^\dagger a_1$ and $a_1^\dagger a_2$ commute with $H = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$.

The operators $a_+^\dagger a_+$ and $a_-^\dagger a_-$ commute with each other, and hence with $H = a_+^\dagger a_+ + a_-^\dagger a_- + 1$.

The first commutator to be checked is

$$
\begin{aligned}
[K_1, K_2] &= \frac{\mathrm{i}}{4}\,[\,a_-^\dagger a_+ + a_+^\dagger a_- \,,\, a_-^\dagger a_+ - a_+^\dagger a_-\,] = \frac{\mathrm{i}}{4}\Big([\,a_+^\dagger a_- \,,\, a_-^\dagger a_+\,] - [\,a_-^\dagger a_+ \,,\, a_+^\dagger a_-\,]\Big) \\
&= \frac{\mathrm{i}}{4}\Big([\,a_+^\dagger a_- \,,\, a_-^\dagger\,]\,a_+ + a_-^\dagger\,[\,a_+^\dagger a_- \,,\, a_+\,] - [\,a_-^\dagger a_+ \,,\, a_+^\dagger\,]\,a_- - a_+^\dagger\,[\,a_-^\dagger a_+ \,,\, a_-\,]\Big) \\
&= \frac{\mathrm{i}}{4}\Big(a_+^\dagger\,[\,a_- \,,\, a_-^\dagger\,]\,a_+ + a_-^\dagger\,[\,a_+^\dagger \,,\, a_+\,]\,a_- - a_-^\dagger\,[\,a_+ \,,\, a_+^\dagger\,]\,a_- - a_+^\dagger\,[\,a_-^\dagger \,,\, a_-\,]\,a_+\Big) \\
&= \frac{\mathrm{i}}{4}\Big(a_+^\dagger a_+ - a_-^\dagger a_- - a_-^\dagger a_- + a_+^\dagger a_+\Big) = \mathrm{i}K_3\ .
\end{aligned}
$$

The two other commutators are computed in the same way.

The fact that K_1, K_2 and K_3 satisfy commutation relations of the angular momentum type implies that $K_3 = L/2$ is half integer: $0\,,\ \pm 1/2\,,\ \pm 1\,,\ \pm 3/2\,,\ \pm 2\,,\ldots$.

Hence L must be integer, as we already noted above.

3f)

$$
\begin{aligned}
\vec{K}^{\,2} &= \frac{1}{4}\Big((a_-^\dagger a_+ + a_+^\dagger a_-)^2 - (a_-^\dagger a_+ - a_+^\dagger a_-)^2 + (a_+^\dagger a_+ - a_-^\dagger a_-)^2\Big) \\
&= \frac{1}{4}\Big((a_-^\dagger a_+)^2 + a_-^\dagger a_+ a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_+ + (a_+^\dagger a_-)^2 \\
&\qquad -(a_-^\dagger a_+)^2 + a_-^\dagger a_+ a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_+ - (a_+^\dagger a_-)^2 + (N_+ - N_-)^2\Big) \\
&= \frac{1}{4}\Big(2a_+ a_+^\dagger a_-^\dagger a_- + 2a_+^\dagger a_+ a_- a_-^\dagger + (N_+ - N_-)^2\Big) \\
&= \frac{1}{4}\Big(2(1 + a_+^\dagger a_+)a_-^\dagger a_- + 2a_+^\dagger a_+(1 + a_-^\dagger a_-) + (N_+ - N_-)^2\Big) \\
&= \frac{1}{4}\Big(2(1 + N_+)N_- + 2N_+(1 + N_-) + N_+^2 - 2N_+N_- + N_-^2\Big) \\
&= \frac{1}{4}\Big(2(N_- + N_+) + (N_+ + N_-)^2\Big) = \frac{N}{2}\left(\frac{N}{2} + 1\right).
\end{aligned}
$$

We know from the theory of angular momentum that the possible eigenvalues of $\vec{K}^{\,2}$ are $k(k+1)$ with $k = 0\,,\ 1/2\,,\ 1\,,\ 3/2\,,\ 2\,,\ldots$. Each value of k defines an irreducible representation of the su(2) Lie algebra defined by the operators K_1, K_2, K_3, and the dimension of this representation is $2k+1$.

The energy eigenvalues of the harmonic oscillator are $E_n = n + 1$, where $n = 0, 1, 2, \ldots$ are the eigenvalues of the number operator $N = N_+ + N_-$. The number operators N_+ and N_- commute, hence they may be quantized simultaneously, and, as we know, they have eigenvalues $n_+ = 0, 1, 2, \ldots$ and $n_- = 0, 1, 2, \ldots$.

Since $n = n_+ + n_-$ it follows that there are $n+1$ orthogonal states with energy $n+1$. If $n = 3$, for example, n_+ may have 4 values $0, 1, 2, 3$. The above formula for $\vec{K}^{\,2}$ shows that all these state vectors are eigenvectors of $\vec{K}^{\,2}$ with $k = n/2$. Since the degeneracy $n+1$ of the energy level is equal to the dimension $2k+1 = n+1$ of the irreducble representation of the Lie algebra, we conclude that all the states with a given energy belong to one single irreducble representation of the Lie algebra.

In other words, the existence of this SU(2) symmetry group for the isotropic two dimensional harmonic oscillator is sufficient to explain the degeneracy of the energy spectrum.

3g) Since p_1 and x_2 commute, and similarly p_2 and x_1, we have that

$$\begin{aligned} H' &= \frac{1}{2m}\,(p_1^{\,2} + 2m\Omega x_2 p_1 + m^2\Omega^2 x_2^2 + p_2^2 - 2m\Omega x_1 p_2 + m^2\Omega^2 x_1^{\,2}) \\ &\qquad + \frac{1}{2}\,m(\omega^2 - \Omega^2)(x_1^{\,2} + x_2^2) \\ &= \frac{1}{2m}\,(p_1^{\,2} + p_2^2) - \Omega(x_1 p_2 - x_2 p_1) + \frac{1}{2}\,m\omega^2(x_1^{\,2} + x_2^2) = H - \Omega L\ . \end{aligned}$$

Using that $H = a_+^\dagger a_+ + a_-^\dagger a_- + 1$ and $L = a_+^\dagger a_+ - a_-^\dagger a_-$ we get that

$$H' = (1-\Omega)\,a_+^\dagger a_+ + (1+\Omega)\,a_-^\dagger a_- + 1\ .$$

The energy eigenvalues are

$$E = n_+(1-\Omega) + n_-(1+\Omega) + 1 \qquad \text{with} \qquad n_+, n_- = 0, 1, 2, \ldots\ .$$

The energy level $E = n+1$ for $\Omega = 0$ is split when $\Omega \neq 0$ so that there is no degeneracy left.

Only the third su(2) generator $K_3 = L/2$ commutes with H', hence the symmetry group is reduced from SU(2) to the two dimensional rotation group SO(2).

3h) The Foucault pendulum rotates because the Earth rotates, of course. We may blame the Coriolis force which makes linear motion deviate towards the right on the northern hemisphere. What more is there to say?

The Hamiltonian

$$H' = \hbar(\omega-\Omega)\,a_+^\dagger a_+ + \hbar(\omega+\Omega)\,a_-^\dagger a_- + \hbar\omega\ ,$$

with $\hbar$ and the oscillation frequency ω made visible again, suggests an unusual way to understand the rotation of the oscillation plane of the pendulum.

We see from H' that the two independent periodic motions of the pendulum, in the reference frame rotating with an angular frequency Ω, are an anticlockwise circular oscillation with angular frequency $\omega-\Omega$ and a clockwise circular oscillation with angular frequency $\omega+\Omega$. The most general classical motion is a superposition of the two circular oscillations,

$$\begin{aligned} x_1(t) &= A\cos((\omega-\Omega)t + 2\alpha) + B\cos((\omega+\Omega)t + 2\beta)\ , \\ x_2(t) &= A\sin((\omega-\Omega)t + 2\alpha) - B\sin((\omega+\Omega)t + 2\beta)\ , \end{aligned}$$

with arbitrary amplitudes A, B and arbitrary phases α, β. The best possible approximation to a linear oscillation is when $A = B$, then

$$\begin{aligned} x_1(t) &= 2A\cos(\Omega t - \alpha + \beta)\cos(\omega t + \alpha + \beta)\ , \\ x_2(t) &= -2A\sin(\Omega t - \alpha + \beta)\cos(\omega t + \alpha + \beta)\ . \end{aligned}$$

When $\omega \gg \Omega$ it seems reasonable to describe this motion as an oscillation with angular frequency ω in a plane rotating in the clockwise direction with angular frequency Ω.