Eksamination FY8104 Symmetry in physics
Tuesday December 13, 2011
Solutions

1a) The group SO(4) consists of the real orthogonal 4×4 matrices with determinant one. A matrix A is orthogonal when its transpose is equal to its inverse, $A^\top = A^{-1}$.

Equivalently, $A \in \mathrm{SO}(4)$ is invertible and preserves the Euclidean metric

$$x^\top x = (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 \qquad \text{where} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

so that if $y = Ax$ we have $y^\top y = x^\top A^\top A x = x^\top x$.

When this holds for all x, it follows that $A^\top A = I =$ the identity matrix.

In a similar way, $B \in \mathrm{SO}(1,3)$ is a real 4×4 matrix which is invertible and preserves the Minkowski metric

$$x^\top G x = (x_1)^2 - (x_2)^2 - (x_3)^2 - (x_4)^2 \qquad \text{where} \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

so that if $y = Bx$ we have $y^\top G y = x^\top B^\top G B x = x^\top G x$.

When this holds for all x, it follows that $B^\top G B = G$, or equivalently $G^{-1} B^\top G = B^{-1}$.

1b) The Lie algebra so(4) consists of the real 4×4 matrices X such that $A = I + \epsilon X \in \mathrm{SO}(4)$ for an infinitesimal real ϵ. The orthogonality condition

$$A^\top = I + \epsilon X^\top = A^{-1} = I - \epsilon X$$

means that X must be antisymmetric, $X^\top = -X$. The most general form is

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

The Lie algebra so(1,3) consists of the matrices Y such that $B = I + \epsilon Y \in \mathrm{SO}(1,3)$ for ϵ infinitesimal. The condition

$$B^\top G B = (I + \epsilon Y^\top) G (I + \epsilon Y) = G$$

means that $Y^\top G + G Y = 0$. With G symmetric, $G^\top = G$, it means that $Z = GY$ must be antisymmetric, $Z^\top = -Z$. The most general form is

$$Y = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix}.$$

2a) Noether's theorem says that a conservation law for a quantum mechanical system is equivalent to a continuous symmetry of the system.

It is a condition that the observable X which is conserved should not be explicitly time dependent. X is then the infinitesimal generator of the continuous symmetry.

2b) Any symmetry of the system transforms an energy eigenstate into an energy eigenstate with the same energy.

It follows that the existence of a symmetry implies the existence of a complete set of energy eigenstates that can be classified into irreducible representations of the symmetry group. For example, the energy eigenstates of the hydrogen atom can be classified by the orbital angular momentum ℓ which labels the irreducible representations of the three dimensional rotation group. All the energy eigenstates of one irreducible representation must have the same energy, by Schur's lemma.

Every non-Abelian group has some irreducible representation of dimension higher than one. Therefore the existence of a non-Abelian symmetry group implies degeneracy in the energy spectrum.

3a) Every representation of a finite group is equivalent to a unitary matrix representation $g \mapsto U(g)$ with $(U(g))^\dagger = (U(g))^{-1} = U(g^{-1})$. For the character value $\chi(g) = \operatorname{Tr} U(g)$ this implies that

$$\chi(g^{-1}) = \operatorname{Tr} U(g^{-1}) = \operatorname{Tr}(U(g))^\dagger = (\operatorname{Tr} U(g))^* = (\chi(g))^* \ .$$

If g and g^{-1} are conjugate group elements, then they have the same character value, which must be real:

$$\chi(g) = \chi(g^{-1}) = (\chi(g))^* \ .$$

An n dimensional matrix $U(g)$ can always be diagonalized, with eigenvalues x_i, $i = 1, 2, \ldots, n$. Then

$$\chi(g) = \operatorname{Tr} U(g) = \sum_{i=1}^{n} x_i \ . \tag{1}$$

If $g^m = I$ = the identity group element, then $(U(g))^m = U(g^m) = I$ = the identity matrix, and hence $(x_i)^m = 1$ for $i = 1, 2, \ldots, n$.

If $m = 2$, then every $x_i = \pm 1$, and $\chi(g)$ is an integer.

If $m = 3$, then either $x_i = 1$ or

$$x_i = \frac{-1 \pm \mathrm{i}\sqrt{3}}{2} \ .$$

Clearly the sum in equation (1) is then an integer if it is real.

If $m = 4$, then either $x_i = \pm 1$ or $x_i = \pm \mathrm{i}$. Clearly the sum in equation (1) is again an integer if it is real.

3b) This is the complete character table:

	$C_1(1)$	$C_2(15)$	$C_3(20)$	$C_4(12)$	$C_5(12)$
$\chi^{(a)}$	1	1	1	1	1
$\chi^{(b)}$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\chi^{(c)}$	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\chi^{(d)}$	4	0	1	-1	-1
$\chi^{(e)}$	5	1	-1	0	0

The sum of squares of the dimensions must be 60, this determines uniquely the dimensions 4 and 5.

The theorem from 3a) implies that the charachter values for the conjugation classes C_2 and C_3 must be integers. The sum of squares must be $60/15 = 4$ for C_2 and $60/20 = 3$ for C_3. This, and the orthogonality of columns, determines uniquely all character values for C_2 and C_3.

The sum over all group elements of the absolute squares of the character values must be 60, this fixes $\chi^{(e)}(C_4) = \chi^{(e)}(C_5) = 0$.

Finally, the orthogonality of columns fixes $\chi^{(d)}(C_4) = \chi^{(d)}(C_5) = -1$.

3c) The dimension is

$$\chi^{(\ell)}(0) = \lim_{\alpha\to 0} \frac{\sin((\ell+\frac{1}{2})\alpha)}{\sin(\frac{\alpha}{2})} = \lim_{\alpha\to 0} \frac{(\ell+\frac{1}{2})\alpha}{\frac{\alpha}{2}} = 2\ell + 1 \; .$$

We need to calculate the character values

$$\chi^{(\ell)}(\alpha) = \frac{\sin((\ell+\frac{1}{2})\alpha)}{\sin(\frac{\alpha}{2})} \; .$$

for $\ell = 1, 2, 3$ and for $\alpha = 0, 180°, 120°, 72°, 144°$. We find:

$$\chi^{(\ell)}(180°) = \frac{\sin((2\ell+1)\,90°)}{\sin(90°)} = (-1)^\ell \; .$$

$$\chi^{(\ell)}(120°) = \frac{\sin((2\ell+1)\,60°)}{\sin(60°)} = \begin{cases} 0 & \text{for} \quad \ell = 1 \, , \\ -1 & \text{for} \quad \ell = 2 \, , \\ 1 & \text{for} \quad \ell = 3 \, . \end{cases}$$

$$\chi^{(\ell)}(72°) = \frac{\sin((2\ell+1)\,36°)}{\sin(36°)} = \begin{cases} \frac{1+\sqrt{5}}{2} & \text{for} \quad \ell = 1 \, , \\ 0 & \text{for} \quad \ell = 2 \, , \\ \frac{-1-\sqrt{5}}{2} & \text{for} \quad \ell = 3 \, . \end{cases}$$

$$\chi^{(\ell)}(144°) = \frac{\sin((2\ell+1)\,72°)}{\sin(72°)} = \begin{cases} \frac{1-\sqrt{5}}{2} & \text{for} \quad \ell = 1 \, , \\ 0 & \text{for} \quad \ell = 2 \, , \\ \frac{-1+\sqrt{5}}{2} & \text{for} \quad \ell = 3 \, . \end{cases}$$

For example, we have that

$$\chi^{(1)}(72^\circ) = \frac{\sin(108^\circ)}{\sin(36^\circ)} = \sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} = \frac{10+2\sqrt{5}}{\sqrt{80}} = \frac{1+\sqrt{5}}{2} \ .$$

Thus we have the following characters:

	$C_1(1)$	$C_2(15)$	$C_3(20)$	$C_4(12)$	$C_5(12)$
$\chi^{(1)}$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\chi^{(2)}$	5	1	-1	0	0
$\chi^{(3)}$	7	-1	1	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

We see that $\chi^{(1)} = \chi^{(b)}$, $\chi^{(2)} = \chi^{(e)}$, and $\chi^{(3)} = \chi^{(c)} + \chi^{(d)}$.

Thus, the $\ell = 0$, $\ell = 1$, and $\ell = 2$ irreducible representations of SO(3) are also irreducible representations of the "football group" F, whereas the $\ell = 3$ irreducible representation of SO(3) splits into two irreducible representations of F. We see that these are all the irreducible representations of F.

3d) The electrons in the "buckyball" move in a potential having F as a symmetry group. Hence the energy eigenstates may be classified into irreducible representations of F.

One way to try to calculate energy levels would be to use as a zero order approximation the energy eigenstates of electrons moving freely on the sphere, and treat the Coulomb potential from the carbon nuclei as a perturbation. The unperturbed states are then classified by their angular momentum $\ell = 0, 1, 2, 3, \ldots$, and the degeneracies of the unpertubed energy levels are $2\ell + 1 = 1, 3, 5, 7, \ldots$.

As we have just shown, the lowest energy levels with $\ell = 0, 1, 2$ will not be split by the perturbing potential. The 7 degenerate energy eigenstates of the $\ell = 3$ level will split into two levels, one with 3 states and one with 4 states.

This pattern of splitting is exact, it must hold to all orders of the perturbation expansion.