Eksamination in FY8104/FY3105 Symmetry in physics
Monday December 9, 2013
Solutions

1a) The centre $Z(G)$ consists of the even powers of a, they commute with p and hence with all group elements, therefore they are alone in their conjugation classes. Subgroups:

$$\begin{aligned}
&H_1 = \{e\} \;, \qquad H_2 = \{e, a^4\} = \{e, j\} \;, \qquad H_3 = \{e, a^2, a^4, a^6\} = \{e, b, j, l\} = Z(G) \;, \\
&H_4 = \{e, a, a^2, \ldots, a^7\} = \{e, a, b, \ldots, m\} \;, \\
&H_5 = \{e, p\} \;, \qquad H_6 = \{e, a^4 p\} = \{e, t\} \;, \\
&H_7 = \{e, p, a^4, a^4 p\} = \{e, p, j, t\} = H_2 \otimes H_5 = H_2 \otimes H_6 \;, \\
&H_8 = \{e, a^2 p, a^4, a^6 p\} = \{e, r, r^2, r^3\} = \{e, r, j, v\} \;, \\
&H_9 = \{e, p, a^2, a^2 p, a^4, a^4 p, a^6, a^6 p\} = \{e, p, b, r, j, t, l, v\} = H_3 \otimes H_5 = H_3 \otimes H_6 \;, \\
&H_{10} = \{e, ap, a^6, a^7 p, a^4, a^5 p, a^2, a^3 p\} = \{e, q, q^2, \ldots, q^7\} = \{e, q, l, w, j, u, b, s\} \;, \\
&H_{11} = G \;.
\end{aligned}$$

1b) All subgroups except H_5 and H_6 are normal, they are unions of conjugation classes:

$$\begin{aligned}
&H_2 = C_1 \cup C_3 \;, \qquad H_3 = H_2 \cup C_2 \cup C_4 \;, \qquad H_4 = H_3 \cup C_5 \cup C_6 \;, \\
&H_7 = H_2 \cup C_7 \;, \qquad H_8 = H_2 \cup C_9 \;, \\
&H_9 = H_3 \cup C_7 \cup C_9 \;, \qquad H_{10} = H_3 \cup C_8 \cup C_{10} \;.
\end{aligned}$$

1c) Every representation of a finite group is equivalent to a unitary matrix representation. When the group element g is represented by a unitary matrix $\mathbf{D}(g)$ the character of g^{-1} is

$$\chi(g^{-1}) = \operatorname{Tr} \mathbf{D}(g^{-1}) = \operatorname{Tr} \mathbf{D}(g)^{-1} = \operatorname{Tr} \mathbf{D}(g)^{\dagger} = (\operatorname{Tr} \mathbf{D}(g))^* = (\chi(g))^* \;.$$

1d) The dimension d of an irreducible representation is a factor of the group order 16, hence $d = 1, 2, 4, \ldots$. We must have $d^2 < 16$, since the sum of squares of the dimensions of all irreducible representations is 16, and there is always the trivial one dimensional representation. Hence $d = 1$ or 2.

1e) There are 10 conjugation classes and 10 irreducible representations. The dimensions are 8 times 1 and 2 times 2, since that makes a square sum of 16.

Let χ be a one dimensional character. Since the group elements a and p generate the whole group G, the character values $\chi(a)$ and $\chi(p)$ determine all other values, so that

$$\chi(a^m p^n) = (\chi(a))^m \, (\chi(p))^n \;.$$

Since $a^8 = p^2 = e$ we must have (in the one dimensional representation) that

$$(\chi(a))^8 = (\chi(p))^2 = 1 \;.$$

Since a and a^5 are conjugates, we must also have that

$$\chi(a) = (\chi(a^5)) = (\chi(a))^5 \;,$$

and hence $(\chi(a))^4 = 1$. The eight one dimensional representations therefore have $\chi(a) = \pm 1, \pm \mathrm{i}$ and $\chi(p) = \pm 1$ in all combinations.

It seems a reaonable guess that the two two dimensional irreducible representations are complex conjugates of each other. Then we can find the two characters by means of the orthogonality relations. Yet another trick is the observation that a^2, a^4, a^6 commute with all of G, hence in an irreducible representation they must be multiples of the identity, by Schur's lemma. Since $(a^2)^4 = e$, the matrix representing a^2 must be λI with $\lambda^4 = 1$.

	C_1 $\{e\}$	C_2 $\{a^2\}$	C_3 $\{a^4\}$	C_4 $\{a^6\}$	C_5 $\{a, a^5\}$	C_6 $\{a^3, a^7\}$	C_7 $\{p, a^4p\}$	C_8 $\{ap, a^5p\}$	C_9 $\{a^2p, a^6p\}$	C_{10} $\{a^3p, a^7p\}$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	1	1	-1	-1	1	-1	1	-1
χ_4	1	1	1	1	-1	-1	-1	1	-1	1
χ_5	1	-1	1	-1	i	$-$i	1	i	-1	$-$i
χ_6	1	-1	1	-1	i	$-$i	-1	$-$i	1	i
χ_7	1	-1	1	-1	$-$i	i	1	$-$i	-1	i
χ_8	1	-1	1	-1	$-$i	i	-1	i	1	$-$i
χ_9	2	2i	-2	-2i	0	0	0	0	0	0
χ_{10}	2	-2i	-2	2i	0	0	0	0	0	0

For any character χ, irreducible or reducible, the set $\{g | \chi(g) = \chi(e)\}$ is a normal subgroup. Thus, χ_2 identifies the normal subgroup H_4, χ_3 identifies H_9, χ_4 identifies H_{10}, χ_5 identifies H_7, χ_6 identifies H_8, χ_7 identifies H_7, and χ_8 identifies H_8. There is no irreducible representation identifying H_2 or H_3, but the reducible character $\chi_3 + \chi_4$ identifies H_3, and $\chi_5 + \chi_6$ identifies H_2.

2a) Take

$$A = \frac{\mathrm{d}}{\mathrm{d}r} + W \qquad \text{with} \qquad W = -\frac{\alpha}{r} + \beta \, .$$

The definition of the Hermitean conjugate operator $A^\dagger$ is that for any two wave functions u, v we have that

$$\int_0^\infty \mathrm{d}r \; (u(r))^* \, (Av(r)) = \int_0^\infty \mathrm{d}r \; (A^\dagger u(r))^* \, v(r) \, .$$

By partial integration, using the boundary conditions on the wave functions u and v, we get that

$$\int_0^\infty \mathrm{d}r \; u^* \, \frac{\mathrm{d}v}{\mathrm{d}r} = u(\infty)^* \, v(\infty) - u(0)^* \, v(0) - \int_0^\infty \mathrm{d}r \; \frac{\mathrm{d}u^*}{\mathrm{d}r} \, v = - \int_0^\infty \mathrm{d}r \; \left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^* \, v \, .$$

This shows that

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^\dagger = -\frac{\mathrm{d}}{\mathrm{d}r} \, .$$

Since $W = W(r)$ is a real function of r, we have that $W^\dagger = W$, and

$$A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}r} + W \; .$$

2b) For any wave function $u = u(r)$ we have that

$$\begin{aligned} A^\dagger A u &= \left(-\frac{\mathrm{d}}{\mathrm{d}r} + W\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} + W\right) u = \left(-\frac{\mathrm{d}}{\mathrm{d}r} + W\right)\left(\frac{\mathrm{d}u}{\mathrm{d}r} + Wu\right) \\ &= -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(-\frac{\mathrm{d}W}{\mathrm{d}r} + W^2\right) u = -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(\frac{\alpha(\alpha-1)}{r^2} - \frac{2\alpha\beta}{r} + \beta^2\right) u \; . \end{aligned} \tag{1}$$

Which shows that

$$A^\dagger A = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\alpha(\alpha-1)}{r^2} - \frac{2\alpha\beta}{r} + \beta^2 \; .$$

To make this resemble the Hamiltonian

$$H = \frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{r^2} - \frac{2}{a_0 r}\right) ,$$

we should choose α as a solution of the equation

$$\alpha(\alpha-1) = \ell(\ell+1) \; , \qquad (\alpha+\ell)(\alpha-\ell-1) = 0 \; ,$$

that is, either $\alpha = -\ell$ or $\alpha = \ell + 1$. We choose $\alpha = \ell + 1$. Why? One good reason is that we want to have $\alpha \neq 0$ when $\ell = 0$. With this choice for α we must choose

$$\beta = \frac{1}{\alpha a_0} = \frac{1}{(\ell+1)a_0} \; .$$

In summary, we choose

$$\alpha_\ell = \ell + 1 \; , \qquad \beta_\ell = \frac{1}{(\ell+1)a_0} \; , \qquad \gamma_\ell = \beta_\ell^2 = \frac{1}{(\ell+1)^2 a_0^2} \; .$$

Then we have that

$$W_\ell(r) = -\frac{\ell+1}{r} + \frac{1}{(\ell+1)a_0} \; ,$$

and the one dimensional ℓ-dependent Hamiltonian may be written as

$$H_\ell = \frac{\hbar^2}{2m}\left(A_\ell^\dagger A_\ell - \gamma_\ell\right) .$$

Now go back to eq. (1), and compute in the same way

$$\begin{aligned} A A^\dagger u &= \left(\frac{\mathrm{d}}{\mathrm{d}r} + W\right)\left(-\frac{\mathrm{d}}{\mathrm{d}r} + W\right) u = \left(\frac{\mathrm{d}}{\mathrm{d}r} + W\right)\left(-\frac{\mathrm{d}u}{\mathrm{d}r} + Wu\right) \\ &= -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(\frac{\mathrm{d}W}{\mathrm{d}r} + W^2\right) u = -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(\frac{\alpha(\alpha+1)}{r^2} - \frac{2\alpha\beta}{r} + \beta^2\right) u \; . \end{aligned}$$

Since we chose $\alpha = \ell + 1$, we get that $\alpha(\alpha+1) = (\ell+1)(\ell+2)$, and

$$H_{\ell+1} = \frac{\hbar^2}{2m}\left(A_\ell A_\ell^\dagger - \gamma_\ell\right) .$$

2c) The equation $A_\ell u = 0$ is this:

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\ell+1}{r} + \frac{1}{(\ell+1)a_0} = 0 \ .$$

It is easily integrated to give that

$$\ln u = (\ell+1)\ln r - \frac{r}{(\ell+1)a_0} + \ln C \ ,$$

where C is an integration constant. That is,

$$u(r) = u_\ell^{(0)}(r) = C\, r^{\ell+1}\, \mathrm{e}^{-r/((\ell+1)a_0)} \ .$$

This is the ground state of H_ℓ because it is an eigenstate of the operator $A_\ell^\dagger A_\ell$ with eigenvalue 0, and an operator of the form $A^\dagger A$ is positive semidefinite: it has only non-negative eigenvalues. Proof: if $A^\dagger A|\psi\rangle = \lambda|\psi\rangle$ and $\langle\psi|\psi\rangle = 1$, then

$$\lambda = \langle\psi|A^\dagger A|\psi\rangle = \langle\phi|\phi\rangle \geq 0 \qquad \text{with} \qquad |\phi\rangle = A|\psi\rangle \ .$$

The energy in this state is

$$E = -\frac{\hbar^2}{2m}\,\gamma_\ell = -\frac{1}{(\ell+1)^2}\,\frac{\hbar^2}{2ma_0^2} \ .$$

The equation $A_\ell^\dagger u = 0$ is this:

$$-\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\ell+1}{r} + \frac{1}{(\ell+1)a_0} = 0 \ .$$

It has the solution

$$u(r) = C\,\frac{1}{r^{\ell+1}}\,\mathrm{e}^{r/((\ell+1)a_0)} \ .$$

This is not an acceptable wave function, because it is infinite in both limits $r \to 0$ and $r \to \infty$. Therefore it is not the ground state of $H_{\ell+1}$.

2d) If $H_\ell u = Eu$ and $v = A_\ell u$, then

$$H_{\ell+1}v = -\frac{\hbar^2}{2m}\left(A_\ell A_\ell^\dagger - \gamma_\ell\right)A_\ell u = A_\ell\left(-\frac{\hbar^2}{2m}\left(A_\ell^\dagger A_\ell - \gamma_\ell\right)\right)u = A_\ell H_\ell u = A_\ell E u = Ev \ .$$

If u is the ground state of H_ℓ, then $A_\ell u = 0$, and there is no eigenstate of $H_{\ell+1}$ with energy E.

In the same way, if $H_{\ell+1}v = Ev$ and $w = A_\ell^\dagger v$, then

$$H_\ell w = -\frac{\hbar^2}{2m}\left(A_\ell^\dagger A_\ell - \gamma_\ell\right)A_\ell^\dagger v = A_\ell^\dagger\left(-\frac{\hbar^2}{2m}\left(A_\ell A_\ell^\dagger - \gamma_\ell\right)\right)v = A_\ell^\dagger H_{\ell+1} v = A_\ell^\dagger E v = Ew \ .$$

2e) We have found energy eigenvalues

$$E_n = -\frac{1}{n^2}\,\frac{\hbar^2}{2ma_0^2} \qquad \text{with} \qquad n = 1, 2, \dots\ .$$

And we have found one eigenstate of H_ℓ with energy E_n for every $\ell = n-1, n-2, \dots, 1, 0$. For a given value of ℓ and a given eigenstate of H_ℓ there are $2\ell+1$ states of the three dimensional hydrogen atom with L_z quantized to $m\hbar$ and $m = -\ell, -\ell+1, \dots, \ell$. Thus the total degeneracy of the energy level E_n is

$$d_n = \sum_{\ell=0}^{n-1}(2\ell+1) = 1 + 3 + 5 + \cdots + (2n-1) = n^2\ .$$

3a) The non-zero commutators are:

$$\begin{aligned}
[\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2] &= \boldsymbol{\lambda}_3\ , & [\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3] &= \boldsymbol{\lambda}_1\ , & [\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_1] &= \boldsymbol{\lambda}_2\ , \\
[\boldsymbol{\lambda}_1, \boldsymbol{\kappa}_2] &= \boldsymbol{\kappa}_3\ , & [\boldsymbol{\lambda}_2, \boldsymbol{\kappa}_3] &= \boldsymbol{\kappa}_1\ , & [\boldsymbol{\lambda}_3, \boldsymbol{\kappa}_1] &= \boldsymbol{\kappa}_2\ , \\
[\boldsymbol{\kappa}_1, \boldsymbol{\lambda}_2] &= \boldsymbol{\kappa}_3\ , & [\boldsymbol{\kappa}_2, \boldsymbol{\lambda}_3] &= \boldsymbol{\kappa}_1\ , & [\boldsymbol{\kappa}_3, \boldsymbol{\lambda}_1] &= \boldsymbol{\kappa}_2\ , \\
[\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2] &= -\boldsymbol{\lambda}_3\ , & [\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_3] &= -\boldsymbol{\lambda}_1\ , & [\boldsymbol{\kappa}_3, \boldsymbol{\kappa}_1] &= -\boldsymbol{\lambda}_2\ .
\end{aligned}$$

3b) Since $\boldsymbol{\lambda}_1^3 = -\boldsymbol{\lambda}_1$ we have that

$$\begin{aligned}
\exp(\alpha\boldsymbol{\lambda}_1) &= \mathbf{I} + \alpha\boldsymbol{\lambda}_1 + \frac{1}{2!}(\alpha\boldsymbol{\lambda}_1)^2 + \frac{1}{3!}(\alpha\boldsymbol{\lambda}_1)^3 + \cdots \\
&= \mathbf{I} + \left(\alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 - \cdots\right)\boldsymbol{\lambda}_1 + \left(\frac{1}{2!}\alpha^2 - \frac{1}{4!}\alpha^4 + \cdots\right)\boldsymbol{\lambda}_1^2 \\
&= \mathbf{I} + \sin\alpha\,\boldsymbol{\lambda}_1 + (1-\cos\alpha)\,\boldsymbol{\lambda}_1^2 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.
\end{aligned}$$

And since

$$\begin{aligned}
\cos(\mathrm{i}\chi) &= \frac{\mathrm{e}^{\mathrm{i}(\mathrm{i}\chi)} + \mathrm{e}^{-\mathrm{i}(\mathrm{i}\chi)}}{2} = \frac{\mathrm{e}^{-\chi} + \mathrm{e}^{\chi}}{2} = \cosh\chi\ , \\
\sin(\mathrm{i}\chi) &= \frac{\mathrm{e}^{\mathrm{i}(\mathrm{i}\chi)} - \mathrm{e}^{-\mathrm{i}(\mathrm{i}\chi)}}{2\mathrm{i}} = \frac{\mathrm{e}^{-\chi} - \mathrm{e}^{\chi}}{2\mathrm{i}} = \mathrm{i}\sinh\chi\ ,
\end{aligned}$$

we have that

$$\exp(\chi\boldsymbol{\kappa}_1) = \exp(\mathrm{i}\chi\boldsymbol{\lambda}_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\mathrm{i}\chi) & -\sin(\mathrm{i}\chi) \\ 0 & \sin(\mathrm{i}\chi) & \cos(\mathrm{i}\chi) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\chi & -\mathrm{i}\sinh\chi \\ 0 & \mathrm{i}\sinh\chi & \cosh\chi \end{pmatrix}.$$

The physical interpretation is that $\exp(\alpha\boldsymbol{\lambda}_1)$ is a rotation by an angle α about the 1 axis (the x axis), whereas $\exp(\chi\boldsymbol{\kappa}_1)$ is a Lorentz boost along the 1 (or x) axis.

Let us elaborate on the last point. The Lorentz boost transforms the coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ to $\widetilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$, where

$$\widetilde{x}^0 = \gamma(x^0 - \beta x^1)\ , \quad \widetilde{x}^1 = \gamma(x^1 - \beta x^0)\ , \quad \widetilde{x}^2 = x^2\ , \quad \widetilde{x}^3 = x^3\ .$$

The velocity between the two reference systems is $c\beta$, and $\gamma = 1/\sqrt{1-\beta^2}$.

The electromagnetic field tensor

$$cF^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & c0 \end{pmatrix}$$

transforms as follows,

$$\widetilde{F}^{\mu\nu} = \Lambda^{\mu}_{\ \rho}\,\Lambda^{\nu}_{\ \sigma}\,F^{\rho\sigma}\ .$$

This gives that

$$\begin{aligned}
\widetilde{E}_x &= c\widetilde{F}^{10} = \Lambda^{1}_{\ \rho}\,\Lambda^{0}_{\ \sigma}\,cF^{\rho\sigma} = \Lambda^{1}_{\ 1}\,\Lambda^{0}_{\ 0}\,cF^{10} + \Lambda^{1}_{\ 0}\,\Lambda^{0}_{\ 1}\,cF^{01} = \gamma^2(1-\beta^2)E_x = E_x\ ,\\
\widetilde{E}_y &= c\widetilde{F}^{20} = \Lambda^{2}_{\ \rho}\,\Lambda^{0}_{\ \sigma}\,cF^{\rho\sigma} = \Lambda^{2}_{\ 2}\,(\Lambda^{0}_{\ 0}\,cF^{20} + \Lambda^{0}_{\ 1}\,cF^{21}) = \gamma\,(E_y - \beta cB_z)\ ,\\
\widetilde{E}_z &= c\widetilde{F}^{30} = \Lambda^{3}_{\ \rho}\,\Lambda^{0}_{\ \sigma}\,cF^{\rho\sigma} = \Lambda^{3}_{\ 3}\,(\Lambda^{0}_{\ 0}\,cF^{30} + \Lambda^{0}_{\ 1}\,cF^{31}) = \gamma\,(E_z + \beta cB_y)\ ,\\
c\widetilde{B}_x &= c\widetilde{F}^{32} = \Lambda^{3}_{\ \rho}\,\Lambda^{2}_{\ \sigma}\,cF^{\rho\sigma} = \Lambda^{3}_{\ 3}\,\Lambda^{2}_{\ 2}\,cF^{32} = cF^{32} = cB_x\ ,\\
c\widetilde{B}_y &= c\widetilde{F}^{13} = \Lambda^{1}_{\ \rho}\,\Lambda^{3}_{\ \sigma}\,cF^{\rho\sigma} = \Lambda^{3}_{\ 3}\,(\Lambda^{1}_{\ 1}\,cF^{13} + \Lambda^{1}_{\ 0}\,cF^{03}) = \gamma\,(cB_y + \beta E_z)\ ,\\
c\widetilde{B}_z &= c\widetilde{F}^{21} = \Lambda^{2}_{\ \rho}\,\Lambda^{1}_{\ \sigma}\,cF^{\rho\sigma} = \Lambda^{2}_{\ 2}\,(\Lambda^{1}_{\ 1}\,cF^{21} + \Lambda^{1}_{\ 0}\,cF^{20}) = \gamma\,(cB_z - \beta E_y)\ .
\end{aligned}$$

We may write the same transformation more compactly in matrix form:

$$\begin{pmatrix} c\widetilde{B}_x + \mathrm{i}\widetilde{E}_x \\ c\widetilde{B}_y + \mathrm{i}\widetilde{E}_y \\ c\widetilde{B}_z + \mathrm{i}\widetilde{E}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & -\mathrm{i}\gamma\beta \\ 0 & \mathrm{i}\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} cB_x + \mathrm{i}E_x \\ cB_y + \mathrm{i}E_y \\ cB_z + \mathrm{i}E_z \end{pmatrix}.$$

The transformation matrix is $\exp(\chi\boldsymbol{\kappa}_1)$ with $\cosh\chi = \gamma$, $\sinh\chi = \gamma\beta$.

The parameter χ is called rapidity.

3c) The complex scalar product $\vec{z}\cdot\vec{z} = \mathbf{z}^\top\mathbf{z}$ is invariant because

$$\mathbf{C}^\top = \exp(\mathbf{L}^\top) = \exp(-\mathbf{L}) = \mathbf{C}^{-1}\ ,$$

and

$$(\mathbf{Cz})^\top(\mathbf{Cz}) = \mathbf{z}^\top\mathbf{C}^\top\mathbf{C}\mathbf{z} = \mathbf{z}^\top\mathbf{z}\ .$$

3d) With $\vec{G} = c\vec{B} + \mathrm{i}\vec{E}$ the complex Lorentz invariant is

$$\vec{G}\cdot\vec{G} = c^2\vec{B}^2 - \vec{E}^2 + 2c\,\mathrm{i}\,\vec{B}\cdot\vec{E}\ .$$

The real part $c^2\vec{B}^2-\vec{E}^2$ and the imaginary part $\vec{B}\cdot\vec{E}$ are both invariant under continuous Lorentz transformations.

The Lorentz force $\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$ on a point charge q moving with velocity $\vec{v}$ changes sign, $\vec{F}\mapsto-\vec{F}$, under a parity inversion. Since $\vec{v}\mapsto-\vec{v}$ under parity, this means that $\vec{E}\mapsto-\vec{E}$ and $\vec{B}\mapsto\vec{B}$. Hence $c^2\vec{B}^2-\vec{E}^2$ is parity invariant, but $\vec{B}\cdot\vec{E}$ is not parity invariant.

The two invariants are (up to factors) $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon^{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu}$ where $\epsilon^{\kappa\lambda\mu\nu}$ is the totally antisymmetric Levi–Civita symbol.