Eksamination in FY8104/FY3105 Symmetry in physics
Tuesday December 15, 2015
Solutions

1a) $\mathbf{M}_{\text{ab}}$ belongs to O(3) because it is an orthogonal matrix, that is, $(\mathbf{M}_{\text{ab}})^{-1} = (\mathbf{M}_{\text{ab}})^T$. It does not belong to SO(3), because its determinant is -1 and not $+1$.

$\mathbf{M}_{\text{ab}}$ leaves the plane $z = -y$ invariant, it is a reflection about this plane, which goes through the points $\mathbf{c}$ and $\mathbf{d}$ and is perpendicular to the line between $\mathbf{a}$ and $\mathbf{b}$.

The matrices doing the interchanges $\mathbf{a} \leftrightarrow \mathbf{c}$, $\mathbf{a} \leftrightarrow \mathbf{d}$, $\mathbf{b} \leftrightarrow \mathbf{c}$, $\mathbf{b} \leftrightarrow \mathbf{d}$, and $\mathbf{c} \leftrightarrow \mathbf{d}$ are

$$\mathbf{M}_{\text{ac}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_{\text{ad}} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{M}_{\text{bc}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathbf{M}_{\text{bd}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_{\text{cd}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1b) The matrix

$$\mathbf{M}_{\text{abc}} = \mathbf{M}_{\text{ab}}\mathbf{M}_{\text{bc}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

is the cyclic permutation $\mathbf{a} \mapsto \mathbf{b} \mapsto \mathbf{c} \mapsto \mathbf{a}$ and $\mathbf{d} \mapsto \mathbf{d}$, which has order 3, that is,

$$(\mathbf{M}_{\text{abc}})^3 = \mathbf{I} = \text{the identity matrix} .$$

It is a rotation by 120°, since it has order 3. It is a rotation and not a reflection, since

$$\det \mathbf{M}_{\text{abc}} = (\det \mathbf{M}_{\text{ab}})(\det \mathbf{M}_{\text{bc}}) = (-1)(-1) = +1 .$$

The rotation axis is $-\mathbf{d}$. We may also describe it as a rotation by 240° about the rotation axis $+\mathbf{d}$.

The matrix

$$\mathbf{M}_{\text{abcd}} = \mathbf{M}_{\text{ab}}\mathbf{M}_{\text{bc}}\mathbf{M}_{\text{cd}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

is the cyclic permutation $\mathbf{a} \mapsto \mathbf{b} \mapsto \mathbf{c} \mapsto \mathbf{d} \mapsto \mathbf{a}$, which has order 4, in fact,

$$(\mathbf{M}_{\text{abcd}})^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad (\mathbf{M}_{\text{abcd}})^4 = \mathbf{I} .$$

We may describe it as a composite transformation $\mathbf{M}_{\text{abcd}} = \mathbf{AB} = \mathbf{BA}$ where $\mathbf{A}$ is a rotation by 90° about the y axis and $\mathbf{B}$ is a reflection about the xz plane,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1c) S_4 has five conjugation classe. 1^4 contains the identity transformation, which is a product of four 1-cycles. 2^2 contains the three permutations that are products of two 2-cycles, for example (12)(34). 31 contains the 3-cycles, for example (123)(4). 21^2 contains the 2-cycles, for example (12)(3)(4). 4 contains the 4-cycles, for example (1234).

The character χ of our representation is:

$$\chi(1^4) = \operatorname{Tr}\mathbf{I} = 3\ , \quad \chi(2^2) = \operatorname{Tr}(\mathbf{M}_{\text{ab}}\mathbf{M}_{\text{cd}}) = \operatorname{Tr}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} = -1\ ,$$

$$\chi(31) = \operatorname{Tr}\mathbf{M}_{\text{abc}} = 0\ , \quad \chi(21^2) = \operatorname{Tr}\mathbf{M}_{\text{ab}} = 1\ , \quad \chi(4) = \operatorname{Tr}\mathbf{M}_{\text{ab}} = -1\ .$$

We recognize this as χ_2 in the table, thus we have an irreducible representation.

If we did not know the character table we would compute

$$\sum_{p\in S_4} |\chi(p)|^2 = 3^2 + 3\times(-1)^2 + 8\times 0^2 + 6\times 1^2 + 6\times(-1)^2 = 24\ .$$

We get the order of the group, this proves that the representation is irreducible.

1d) How many ways can we permute the corners of the cube?

The first corner we may move to any one of eight positions. This corner has three neighbours. There are three possible positions for the first neighbouring corner, and then two possible positions for the second neighbouring corner. That is all the freedom we have. Thus the total number of possible symmetry transformations is

$$8\times 3\times 2 = 48\ .$$

The symmetry group of the cube has twice as many elements as the symmetry group of the tetrahedron.

One obvious symmetry (symmetry transformation) of the cube which is not a symmetry of the tetrahedron is the space inversion $-\mathbf{I}$, which commutes with all 3×3 matrices. For every symmetry $\mathbf{R}$ of the tetrahedron we have two symmetries $\pm\mathbf{R}$ of the cube, and $-\mathbf{R}$ is not a symmetry of the tetrahedron. Just by counting, we know that these are all the symmetries of the cube.

To get the character table of the symmetry group of the cube we double the character table of the symmetry group of the tetrahedron, as follows.

	1 1^4	3 2^2	8 31	6 21^2	6 4	1 -1^4	3 -2^2	8 -31	6 -21^2	6 -4
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	3	-1	0	1	-1	3	-1	0	1	-1
χ_3	2	2	-1	0	0	2	2	-1	0	0
χ_4	3	-1	0	-1	1	3	-1	0	-1	1
χ_5	1	1	1	-1	-1	1	1	1	-1	-1
χ_6	1	1	1	1	1	-1	-1	-1	-1	-1
χ_7	3	-1	0	1	-1	-3	1	0	-1	1
χ_8	2	2	-1	0	0	-2	-2	1	0	0
χ_9	3	-1	0	-1	1	-3	1	0	1	-1
χ_{10}	1	1	1	-1	-1	-1	-1	-1	1	1

2a) We need examples of permutations belonging to the different conjugation classes. Note that we calculated the character of the representation $\mathbf{D}_\beta$ in problem 1c).

- 2^2: $(12)(34) = T_1T_3$. Representation matrices:

$$\mathbf{D}_\alpha(T_1T_3) = \mathbf{D}_\alpha(T_1)\mathbf{D}_\alpha(T_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{D}_\beta(T_1T_3) = \mathbf{D}_\beta(T_1)\mathbf{D}_\beta(T_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

- 31: $(123)(4) = T_1T_2$. Representation matrices:

$$\mathbf{D}_\alpha(T_1T_2) = \mathbf{D}_\alpha(T_1)\mathbf{D}_\alpha(T_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{D}_\beta(T_1T_2) = \mathbf{D}_\beta(T_1)\mathbf{D}_\beta(T_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

- 21^2: $(12)(3)(4) = T_1$. Representation matrices: $\mathbf{D}_\alpha(T_1)$ and $\mathbf{D}_\beta(T_1)$.
- 4: $(1234) = T_1T_2T_3$. Representation matrices:

$$\mathbf{D}_\alpha(T_1T_2T_3) = \mathbf{D}_\alpha(T_1T_2)\mathbf{D}_\alpha(T_3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{D}_\beta(T_1T_2T_3) = \mathbf{D}_\beta(T_1T_2)\mathbf{D}_\beta(T_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

By tracing these matrices we get the characters:

	1^4	2^2	31	21^2	4
χ_α	4	0	1	2	0
χ_β	3	-1	0	1	-1

We see rather directly that $\chi_\alpha = \chi_1 + \chi_2$ and $\chi_\beta = \chi_2$.

We calculate the character χ of the tensor product representation $\mathbf{D} = \mathbf{D}_\alpha \otimes \mathbf{D}_\beta$ in the following way,

$$\begin{aligned} \chi(p) &= \operatorname{Tr} \mathbf{D}(p) = \operatorname{Tr}\big(\mathbf{D}_\alpha(p) \otimes \mathbf{D}_\beta(p)\big) = \big(\operatorname{Tr} \mathbf{D}_\alpha(p)\big)\big(\operatorname{Tr} \mathbf{D}_\beta(p)\big) = \chi_\alpha(p)\chi_\beta(p) \\ &= \big(\chi_1(p) + \chi_2(p)\big)\,\chi_2(p) = \big(1 + \chi_2(p)\big)\,\chi_2(p) = \chi_2(p) + [\chi_2(p)]^2 . \end{aligned}$$

We need to decompose the character $\chi_\gamma = (\chi_2)^2$ into irreducible characters. It is:

	1^4	2^2	31	21^2	4
χ_γ	9	1	0	1	1

Its scalar product with itself is

$$9^2 + 3 \times 1^2 + 8 \times 0^2 + 6 \times 1^2 + 6 \times 1^2 = 96 = 4 \times 24 \ .$$

Hence the sum of squares of multiplicities of irreducible representations is four. Since a representation of dimension nine (an odd number) can not be the direct sum of two identical representations, it must contain four irreducible representations, each with multiplicity one. We see rather directly that

$$\chi_\gamma = \chi_1 + \chi_2 + \chi_3 + \chi_4 \ .$$

We could also calculate scalar products, for example

$$\begin{aligned} \langle \chi_3, \chi_\gamma \rangle &= \sum_{p \in S_4} (\chi_3(p))^* \chi_\gamma(p) \\ &= 2 \times 9 + 3 \times 2 \times 1 + 8 \times (-1) \times 0 + 6 \times 0 \times 1 + 6 \times 0 \times 1 = 1 \times 24 \ , \end{aligned}$$

which shows that χ_γ contains χ_3 with multiplicity one.

Altogether, $\chi = \chi_1 + 2\chi_2 + \chi_3 + \chi_4$, which shows that the representation $\mathbf{D} = \mathbf{D}_\alpha \otimes \mathbf{D}_\beta$ of S_4 contains five irreducible representations, one of them twice.

2b) Remember that we are studying how small deformations of the methane molecule transform under the rotations and reflections that are symmetries of the undeformed molecule. It is now a question of how to interpret physically the decomposition into irreducible representations.

We study how the four hydrogen atoms move, but we do not consider translations of the whole molecule such that it does not change its shape. Then we may forget about the motion of the carbon atom, because that is uniquely given by the requirement that the centre of mass should not move.

The molecule may rotate without changing shape, this corresponds to one three dimensional irreducible representation, with character either χ_2 or χ_4. Actually χ_4 (we do not go into here how we know).

Since we consider electromagnetic radiation with wave lengths much larger than the size of the molecule, the radiation is mainly due to oscillating electric dipole moments. The methane molecule in its ground state is too symmetric to have a nonzero electric dipole moment, therefore there is little radiation from rotation without change of shape.

There remain four irreducible representation that define four vibration modes with four different frequencies.

One of the vibration modes is invariant under the S_4 group of transformations, it corresponds to the representation χ_1. This is a breathing mode, or pulsation mode, in which the molecule just expands and shrinks in size. Since this mode is also too symmetric to have an electric dipole moment, it must radiate rather weakly.

We do not go more deeply into this subject here.

3a) The special four-momentum

$$\mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is invariant under the Lorentz transformation $\exp(\alpha_1 \boldsymbol{\mu}_1 + \alpha_2 \boldsymbol{\mu}_2 + \alpha_3 \boldsymbol{\lambda}_3)$ because

$$\begin{aligned}
\boldsymbol{\mu}_1 \, \mathbf{k} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \\
\boldsymbol{\mu}_2 \, \mathbf{k} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \\
\boldsymbol{\lambda}_3 \, \mathbf{k} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .
\end{aligned}$$

3b)

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad (\boldsymbol{\mu}_1)^2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} , \qquad (\boldsymbol{\mu}_1)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Hence,

$$\exp(\alpha \boldsymbol{\mu}_1) = \mathbf{I} + \alpha \boldsymbol{\mu}_1 + \frac{1}{2} \alpha^2 (\boldsymbol{\mu}_1)^2 = \begin{pmatrix} 1 + \frac{\alpha^2}{2} & 0 & \alpha & -\frac{\alpha^2}{2} \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \frac{\alpha^2}{2} & 0 & \alpha & 1 - \frac{\alpha^2}{2} \end{pmatrix} .$$

And

$$\exp(\alpha \boldsymbol{\mu}_1) \, \mathbf{k} = \begin{pmatrix} 1 + \frac{\alpha^2}{2} & 0 & \alpha & -\frac{\alpha^2}{2} \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & -\alpha \\ \frac{\alpha^2}{2} & 0 & \alpha & 1 - \frac{\alpha^2}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k} .$$

3c)

$$\begin{aligned}
[\boldsymbol{\mu}_1, \boldsymbol{\mu}_2] &= [\boldsymbol{\lambda}_1 + \boldsymbol{\kappa}_2, \boldsymbol{\lambda}_2 - \boldsymbol{\kappa}_1] = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2] - [\boldsymbol{\lambda}_1, \boldsymbol{\kappa}_1] + [\boldsymbol{\kappa}_2, \boldsymbol{\lambda}_2] - [\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1] \\
&= \boldsymbol{\lambda}_3 - 0 + 0 - \boldsymbol{\lambda}_3 = 0 , \\
[\boldsymbol{\lambda}_3, \boldsymbol{\mu}_1] &= [\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_1] + [\boldsymbol{\lambda}_3, \boldsymbol{\kappa}_2] = \boldsymbol{\lambda}_2 - \boldsymbol{\kappa}_1 = \boldsymbol{\mu}_2 , \\
[\boldsymbol{\lambda}_3, \boldsymbol{\mu}_2] &= [\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_2] - [\boldsymbol{\lambda}_3, \boldsymbol{\kappa}_1] = -\boldsymbol{\lambda}_1 - \boldsymbol{\kappa}_2 = -\boldsymbol{\mu}_1 .
\end{aligned}$$

3d) From the last two commutation relations we get that

$$[-\mathrm{i}S_3, -\mathrm{i}M_1] = -\mathrm{i}M_2 \ , \qquad [-\mathrm{i}S_3, -\mathrm{i}M_2] = -(-\mathrm{i}M_2) \ ,$$

or,

$$[S_3, M_1] = \mathrm{i}M_2 \ , \qquad [S_3, M_2] = -\mathrm{i}M_1 \ .$$

Hence,

$$\begin{aligned} [S_3, M_+] &= [S_3, M_1] + \mathrm{i}\,[S_3, M_2] = \mathrm{i}M_2 + \mathrm{i}\,(-\mathrm{i}M_1) = M_1 + \mathrm{i}M_2 = M_+ \ , \\ [S_3, M_-] &= [S_3, M_1] - \mathrm{i}\,[S_3, M_2] = \mathrm{i}M_2 - \mathrm{i}\,(-\mathrm{i}M_1) = -M_1 + \mathrm{i}M_2 = -M_- \ . \end{aligned}$$

Now assume that $S_3\,|\sigma\rangle = \sigma\,|\sigma\rangle$ and define $|\psi\rangle = M_+\,|\sigma\rangle$, $|\phi\rangle = M_-\,|\sigma\rangle$. It follows that

$$\begin{aligned} S_3\,|\psi\rangle &= S_3 M_+\,|\sigma\rangle = \big([S_3, M_+] + M_+ S_3\big)\,|\sigma\rangle = (M_+ + M_+\sigma)\,|\sigma\rangle \\ &= (\sigma+1) M_+\,|\sigma\rangle = (\sigma+1)\,|\psi\rangle \ , \\ S_3\,|\phi\rangle &= S_3 M_-\,|\sigma\rangle = \big([S_3, M_-] + M_- S_3\big)\,|\sigma\rangle = (-M_- + M_-\sigma)\,|\sigma\rangle \\ &= (\sigma-1) M_-\,|\sigma\rangle = (\sigma-1)\,|\phi\rangle \ . \end{aligned}$$

Note that these calculations hold even if $|\psi\rangle = 0$ or $|\phi\rangle = 0$. But only nonzero vectors count as eigenvectors, by definition.

3e) Because $M_1 M_2 = M_2 M_1$ we have that

$$\begin{aligned} M_+ M_- &= M_- M_+ = (M_1 - \mathrm{i}M_2)(M_1 + \mathrm{i}M_2) = M_1^2 + M_2^2 + \mathrm{i}\,(M_1 M_2 - M_2 M_1) \\ &= M_1^2 + M_2^2 \ , \end{aligned}$$

and this operator commutes with M_1 and M_2. We only need to show that it commutes with S_3. Proof, by the Leibniz rule:

$$[S_3, M_+ M_-] = [S_3, M_+]\,M_- + M_+\,[S_3, M_-] = M_+ M_- - M_+ M_- = 0 \ .$$

That $M_+ M_-$ must have non-negative eigenvalues follows from the postulate that the scalar product in the Hilbert space should be positive definite. In fact, if $|\lambda\rangle \neq 0$ and $M_+ M_-\,|\lambda\rangle = \lambda\,|\lambda\rangle$, and if we define $|\phi\rangle = M_-\,|\lambda\rangle$, then

$$0 \leq \langle\phi|\phi\rangle = \big(\langle\lambda|M_-^\dagger\big)\big(M_-\,|\lambda\rangle\big) = \langle\lambda|M_+ M_-|\lambda\rangle = \lambda\,\langle\lambda|\lambda\rangle \ .$$

Here $\langle\lambda|\lambda\rangle > 0$, by definition, implying that $\lambda \geq 0$. Hence we may write $\lambda = \rho^2$ with $\rho \geq 0$. We see that if $\lambda = 0$, then $\langle\phi|\phi\rangle = 0$, implying that $|\phi\rangle = M_-\,|\lambda\rangle = 0$. Since $M_+ M_- = M_- M_+$, $\lambda = 0$ implies also that $M_+\,|\lambda\rangle = 0$, by a similar reasoning. If $M_-\,|\lambda\rangle = M_+\,|\lambda\rangle = 0$, then

$$M_1\,|\lambda\rangle = \frac{1}{2}\,(M_+ + M_-)\,|\lambda\rangle = 0 \ , \qquad M_2\,|\lambda\rangle = -\frac{\mathrm{i}}{2}\,(M_+ - M_-)\,|\lambda\rangle = 0 \ .$$

This gives a one dimensional representation of the little group, with

$$M_1\,|0,\sigma\rangle = M_2\,|0,\sigma\rangle = 0 \ , \qquad S_3\,|0,\sigma\rangle = \sigma\,|0,\sigma\rangle \ . \tag{1}$$

Here σ is an arbitrary real number.

If $\rho > 0$, then the irreducible representation of the little group must be infinite dimensional. With normalized eigenvectors $|\rho, \sigma\rangle$ and suitable choices of relative phases of these eigenvectors, we get that

$$S_3 |\rho, \sigma\rangle = \sigma |\rho, \sigma\rangle \ , \qquad M_+ |\rho, \sigma\rangle = \rho |\rho, \sigma + 1\rangle \ , \qquad M_- |\rho, \sigma\rangle = \rho |\rho, \sigma - 1\rangle \ .$$

Photons have $\rho = 0$, $\sigma = \pm 1$. The quantum number σ is called helicity, it is the spin component along the direction of motion of the photon. The two different photon states with $\sigma = +1$ and $\sigma = -1$ are related by a parity transformation.

There are no known massless particles that are described by the infinite dimensional representations of the little group. According to Feynman, there is a rule in physics that everything which is mathematically possible, is compulsory. This case seems to be the exception that confirms the rule.

The existence of an infinite spin particle would be a catastrophy in cosmology, because it would be a system with infinite heat capacity. All the energy of the Universe would be spent on creating such particles.

3f) Straightforward matrix multiplications give that $\mathbf{Q}^2 = \mathbf{I}$, hence $\mathbf{Q}^{-1} = \mathbf{Q}$, and

$$\begin{aligned} \mathbf{Q}\boldsymbol{\lambda}_1\mathbf{Q}^{-1} &= \boldsymbol{\lambda}_1 \ , \qquad \mathbf{Q}\boldsymbol{\lambda}_2\mathbf{Q}^{-1} = -\boldsymbol{\lambda}_2 \ , \qquad \mathbf{Q}\boldsymbol{\lambda}_3\mathbf{Q}^{-1} = -\boldsymbol{\lambda}_3 \ , \\ \mathbf{Q}\boldsymbol{\kappa}_1\mathbf{Q}^{-1} &= -\boldsymbol{\kappa}_1 \ , \qquad \mathbf{Q}\boldsymbol{\kappa}_2\mathbf{Q}^{-1} = \boldsymbol{\kappa}_2 \ , \qquad \mathbf{Q}\boldsymbol{\kappa}_3\mathbf{Q}^{-1} = \boldsymbol{\kappa}_3 \ . \end{aligned}$$

Hence,

$$\mathbf{Q}\boldsymbol{\mu}_1\mathbf{Q}^{-1} = \boldsymbol{\mu}_1 \ , \qquad \mathbf{Q}\boldsymbol{\mu}_2\mathbf{Q}^{-1} = -\boldsymbol{\mu}_2 \ .$$

When we quantize, we should have that

$$QM_1Q^{-1} = M_1 \ , \qquad QM_2Q^{-1} = -M_2 \ , \qquad QS_3Q^{-1} = -S_3 \ .$$

Or equivalently,

$$QM_+Q^{-1} = M_- \ , \qquad QM_-Q^{-1} = M_+ \ , \qquad QS_3Q^{-1} = -S_3 \ .$$

This results from equation (1) if we define

$$Q |\rho, \sigma\rangle = \eta |\rho, -\sigma\rangle \ .$$

For example,

$$\begin{aligned} QM_+Q^{-1} |\rho, \sigma\rangle &= \eta^{-1} QM_+ |\rho, -\sigma\rangle = \eta^{-1} \rho \, Q |\rho, -\sigma + 1\rangle = \eta^{-1} \rho \eta |\rho, \sigma - 1\rangle \\ &= \rho |\rho, \sigma - 1\rangle = M_- |\rho, \sigma\rangle \ . \end{aligned}$$

The intrinsic parity η is an arbitrary phase factor. The matrix identity $\mathbf{Q}^2 = \mathbf{I}$ would correspond to the relation $Q^2 = \eta^2 I$. This is acceptable, since multiplication by an overall phase factor η^2 in quantum mechanics is a physical identity transformation. If we fix $\eta = \pm 1$ then we get that $Q^2 = I$.

Photons have intrinsic parity $\eta = -1$.