Norges teknisk–naturvitenskapelige universitet NTNU

Institutt for fysikk
Fakultet for naturvitenskap og teknologi

Exam in FY3106/FY8303 Phase Transitions and Critical Phenomena

Saturday, June 11, 2016
09:00–13:00

Allowed help: Alternativ C

This problem set consists of 2 pages, plus an Appendix of one page.

Problem 1

As a result of a renormalization group transformation, the free energy has the form

$$\omega(\tau, h) = \omega_0 + \frac{1}{l^d}\ \omega_s(l^{y_\tau}\tau, l^{y_h}h)\ , \tag{1}$$

where $\tau = (T - T_c)/T_c$ is the reduced temperature, h is the magnetic field and l the scale factor.

a) Use this expression to derive the heat capacity exponent α in terms of y_τ.

b) Use the renormalization group to derive the hyperscaling relation $2 - \alpha = d\nu$.

Problem 2

A spin system belonging to the universality class $(d, n) = (2, 2)$ consists of classical spins of length $s = 1$ on a square lattice. The orientation of spin i is defined by its angle ϕ_i with the x-axis. The hamiltonian in zero external magnetic field is

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j)\ , \tag{2}$$

where the sum runs over nearest neighbors. The spin-spin correlation function is defined as

$$\Gamma(\vec{r}) = \langle \vec{s}(\vec{r}) \cdot \vec{s}(\vec{0}) \rangle\ . \tag{3}$$

a) Show that at sufficiently low temperature, the correlation function may be written

$$\Gamma(\vec{r}) = \exp\left[\langle [\phi(\vec{r}) - \phi(\vec{0})]\phi(\vec{0}) \rangle\right]\ . \tag{4}$$

b) For large $r = |\vec{r}|$ we have in two dimensions $\Gamma(\vec{r}) \sim r^{-\eta}$. In the gaussian approximation, we furthermore have that

$$\langle \phi(\vec{r})\phi(\vec{0}) \rangle \sim \text{Const} - \frac{k_b T}{2\pi J}\ \ln\left(\frac{r}{r_0}\right)\ . \tag{5}$$

Use this to determine η.

c) Why is this result for η not valid for all temperatures? Which mechanism destroys the result?

Problem 3

We consider the Ising model on a triangular lattice using the Niemejer-van Leeuwen 3-spin block renormalization group construction with the majority rule. We only consider the first cumulant approximation.

a) Let the initial hamiltonian only contain the nearest neighbor coupling K_1 (with $1/k_B T$ absorbed into it). Show that the renormalization group transformation becomes

$$K_1' = 2\psi^2(K_1)K_1 \;, \tag{6}$$

where

$$\psi(K_1) = \frac{e^{3K_1} + e^{-K_1}}{e^{3K_1} + 3e^{-K_1}} \;. \tag{7}$$

b) Show that one of the fixed points in this transformation is

$$K_1^* = \frac{1}{4}\, \ln[1 + 2\sqrt{2}] \;. \tag{8}$$

Determine the two other fixed points. Which of the fixed points describe the critical point in the model? What is the stability of the fixed points?

c) Determine the exponent y_τ defined in equation (1) using the renormalization group transformation, equation (6), explaining the steps on the way. We give as information that

$$\left(\frac{dK_1'}{dK_1}\right)_{K_1=K_1^*} \approx 1.62352... \;, \tag{9}$$

where K_1^* is the fixed point given in equation (8).

The following information may be of some use:

Heat capacity near a critical point: $c = \partial^2\omega/\partial\tau^2 \sim |\tau|^{-\alpha}$.

Correlation length near a critical point: $\xi \sim |\tau|^{-\nu}$.

Cumulant expansion: $\langle e^{\lambda V}\rangle = \exp\left[\sum_{n=1}^{\infty} \lambda^n C_n\right]$ where C_n is the nth cumulant. $C_1 = \langle V\rangle$, $C_2 = (\langle V^2\rangle - \langle V\rangle^2)/2$ etc.