Institutt for fysikk, NTNU

Takehome Exam FY3106/FY8303 Phase transitions and critical phenomena
12:00 December 2- 12:00 December 9, 2019

Problem 1

The Landau (mean-field) energy functional of phase transitions, is given by

$$E[\phi] = \frac{r}{2}\phi^2 + u\phi^4 + v\phi^6 - h\phi$$

where h is en external uniform magnetic field, and ϕ is an order parameter field which we take here to be scalar and real-valued.

a) Show that when $h = 0$, this theory predicts a continuous phase transition when $u > 0$ and a discontinuous phase transition when $u < 0$.

b) Find the value of $r = r(u, v)$ where the transitions occur.

c) Set $v = 1$ and sketch a phase-diagram in (r, u)-space.

Problem 2

The Ginzburg-Landau energy functional for an n-component ferromagnet, is given by (in the standard notation used in lectures for r and u)

$$E[\mathbf{m}] = \int d^d r \left(\frac{1}{2} (\nabla \mathbf{m})^2 + \frac{r}{2}\mathbf{m}^2 + u\left(\mathbf{m}^2\right)^2 + v\sum_{\alpha=1}^{n} m_\alpha^4 \right)$$

where $\mathbf{m}$ is an n-component vector, $\mathbf{m}^2 = \sum_{i=1}^n m_i m_i$, and $(u > 0, v \geq 0)$.

a) Find, to one-loop order, the recursion relations on differential form for (r, u, v).

b) Find all the fixed points of the recursion relations.

c) Assume that the system is at its critical temperature, $r = r_c(u, v)$. Investigate the flows in (u, v)-space and find the stable fixed points for $n < n_c$ and $n > n_c$, where $n_c = 4 + \mathcal{O}(\epsilon)$). Here, $\epsilon = 4 - d$.

d) Find the temperature exponent for each of these fixed points and deduce which will be observed experimentally.

Problem 3

In lectures, we have considered the RG equations for the overall charge-neutral $2D$ Coulomb gas, and found the following recursion relations for the fugacity $\zeta(l)$ and temperature $\tau(l)$

$$\frac{d\tau}{dl} = \frac{1}{2}\zeta^2$$

$$\frac{d\zeta^2}{dl} = \left(4 - \frac{1}{\tau}\right)\zeta^2$$

These recursion relations describe a low-temperature insulating phase and a high-temperature metallic phase separated by the separatrix $\zeta = -|T(l)|$, where $T(l) \approx 4\tau(l) - 1$.

a) Consider a d-dimensional overall charge-neutral system of N_+ positive point-charges $q_i = +1$ and N_- negative point-charges with charge $q_i = -1$, interacting with the potential

$$V(r) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{(4\pi)^{d/2}} \left[\left(\frac{r}{a}\right)^{2-d} - 1 \right]$$

Derive the recursion relations for $(\zeta(l), T(l))$, draw the flow diagram, and find the possible phases of this system, when $d \geq 3$. Here, a is a short-distance cutoff.

b) Consider a lattice gauge-theory in $d = 3$ dimensions on a simple cubic lattice, with the partition function

$$\begin{aligned} Z &= \int_0^{2\pi} \prod_i d\mathbf{A}_i \; e^{\kappa \sum_{i,\mu} \cos(B_{i,\mu})} \\ B_{i,\mu} &= (\boldsymbol{\Delta} \times \mathbf{A})_{i,\mu} \end{aligned}$$

where Δ_μ denotes a lattice gradient in direction μ, $\Delta_\mu f_i = f_{i+\hat{\mu}} - f_i$, $\boldsymbol{\Delta} \times \mathbf{A}$ is a lattice-curl, and $\hat{\mu}$ is a unit vector in direction μ. Use the Villain-approximation and show that the partition function may be written on the form

$$Z = \sum_{\{b_{i,\mu}\}} e^{-\frac{1}{2\kappa} \sum_{i,\mu} b_{i,\mu}^2}$$

with the constraint

$$\boldsymbol{\Delta} \times \mathbf{b} = 0$$

on each elementary plaquette on the lattice.

c) Solve the constraint and show that the partition function may be expressed as

$$Z = \sum_{\{l_i\}} e^{-\frac{1}{2\kappa} \sum_{i,\mu} (l_{i+\mu} - l_i)^2}$$

where $\{l_i\}$ are integers defined on the dual lattice.

d) Use the Poisson-summation formula, elevate l_i to real-valed fields θ_i, and integrate out the $\{\theta_i\}$-fields. From this, deduce the phase(s) of the lattice gauge-theory given above.