Institutt for fysikk, NTNU

Takehome Exam FY8303 Phase transitions and critical phenomena
09:00 December 3- 18:00 December 4, 2021

Problem 1

A simple model of a superfluid is given by the Hamiltonian

$$H = \frac{\rho}{2} \int d^d r \, (\nabla\theta)^2$$

where ρ is a short-distance phase stiffness and θ is the phase of a complex wave-function Ψ for the particles in the superfluid, $\Psi = |\Psi_0| e^{i\theta}$, and $|\Psi_0|$ is an amplitude which we may regard as spatially independent. The partition function Z for the system is given by (in standard notation)

$$Z = \int D\theta \; e^{-H}$$

a) Use dimensional analysis to determine the scaling dimension of ρ.

b) When $d > 2$, we have two options for defining an order parameter for this system, either a local one or a global one. The local one is given by $\langle \Psi \rangle$, the global one is the phase-stiffness or helicity modulus Υ of the system. Introduce $t = (T - T_c/T_c)$, where T_c is the critical temperature. Close to T_c, we have

$$|\langle \Psi \rangle| \sim |t|^{\beta}$$
$$\Upsilon \sim |t|^{\omega}$$

Here, β and ω are critical exponents of the local and global order-parameters, respectively. It is natural to compare 2β to ω. Explain why.

c) Use the result of **a**) and scaling relation for critical exponent to compute $2\beta - \omega$. Express your answer in terms of one or more of the critical exponents $(\alpha, \beta, \gamma, \delta, \eta, \nu)$.

d) Sketch $|\langle \Psi \rangle|^2$ and Υ close to T_c.

Problem 2

Consider the $3DXY$-model

$$H = -K \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

with $\mathbf{S}_i = (\cos\theta_i, \sin\theta_i)$ in the low-temperature limit, where we may use the spin-wave approximation

$$H = \frac{K}{2} \int d^d r \;\; (\nabla\theta)^2$$

A factor $1/k_B T$ is absorbed in K.

a) Calculate the correlation function $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ in the low-temperature limit.

b) Explain how spin-waves affect long-range spin-correlations in the $3DXY$-model.

Problem 3

Consider a loop-gas (dual) representation of the XY-model

$$H = J \sum_{\langle i,j \rangle} \left[1 - \cos\left(\theta_i - \theta_j\right)\right]$$

In lectures, we have obtained the following representation of the partition function in terms of a gas of integer-valued link-variables $b_\mu(\mathbf{r})$ living on links starting at a point $\mathbf{r}$ and pointing in the direction μ

$$Z = \sum_{\{\mathbf{b}\}} \exp\left(-\frac{1}{2K} \sum_{\mathbf{r}} \mathbf{b}(\mathbf{r})^2\right) \prod_{\mathbf{r}} \delta_{\nabla \cdot \mathbf{b}, 0}$$

where we have defined $K \equiv \beta J$. The constraint $\nabla \cdot \mathbf{b} = 0$ applies at every lattice point and means that the link-currents must form closed loops. This contraint endows this integer-valued Gaussian theory with strong interactions, facilitating a phase-transition.

a) Imagine that the contraint is removed. Compute the partition function for this case, and show that there is no phase-transition.

b) We now include an external magnetic field in the problem, such that the Hamiltonian takes the form

$$H = J \sum_{\langle i,j \rangle} \left[1 - \cos\left(\theta_i - \theta_j\right)\right] - h \sum_i \cos(\theta_i)$$

Use the same techniques as we used in lectures, and derive a loop-gas representation of this model. (Hint: Use Villain approximations on both terms in the Hamiltonian and use two Hubbard-Stratonovic decouplings of the terms involving θ_i).

c) Use the derived loop-gas representation to explain why the presence of a magnetic field destroys the $h = 0$ phase-transition. Also use this loop-gas representation to explain how the phase-transition is restored when $h \to 0$.

Jacobi ϑ_3-function:

$$\vartheta_3(z, \tau) = \sum_{b=-\infty}^{\infty} q^{b^2} \eta^b$$

with $q = \exp(i\pi\tau)$ and $\eta = \exp(2i\pi z)$. This function is analytic in (z, τ).