Institutt for fysikk, NTNU

Exam FY3106/FY8303 Phase transitions and critical phenomena
09:00-13:00 December 1, 2023

Problem 1 (10+10+10+10+10+10=60)

A simple model of a superfluid is given by the Hamiltonian

$$H = \frac{\rho}{2} \int d^d r \, (\nabla\theta)^2$$

where ρ is a short-distance phase stiffness and θ is the phase of a complex wave-function Ψ for the particles in the superfluid, $\Psi = |\Psi_0| e^{i\theta}$, and $|\Psi_0|$ is an amplitude which we may regard as spatially independent. The partition function Z for the system is given by (in standard notation)

$$Z = \int D\theta \; e^{-H}$$

a) Use dimensional analysis to determine the scaling dimension of ρ.

b) When $d > 2$, we have two options for defining an order parameter for this system, either a local one or a global one. The local one is given by $\langle \Psi \rangle$, the global one is the phase-stiffness or helicity modulus Υ (phase-stiffness) of the system. Introduce $t = (T - T_c)/T_c$, where T_c is the critical temperature. Close to T_c, we have

$$\begin{aligned} |\langle \Psi \rangle| &\sim |t|^{\beta} \\ \Upsilon &\sim |t|^{\omega} \end{aligned}$$

Here, β and ω are critical exponents of the local and global order-parameters, respectively. It is natural to compare 2β to ω. Explain why.

c) Use the result of **a**) and scaling relations for critical exponent to compute $2\beta - \omega$. Express your answer only in terms of one or more of the critical exponents $(\alpha, \beta, \gamma, \delta, \eta, \nu)$.

d) Sketch $|\langle \Psi \rangle|^2$ and Υ close to T_c.

e) Consider the $3DXY$-model

$$H = -K \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

with $\mathbf{S}_i = (\cos\theta_i, \sin\theta_i)$. A factor $1/k_BT$ is absorbed in K. Calculate the correlation function $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ in the low-temperature limit.

f) Explain how spin-waves affect long-range spin-correlations in the $3DXY$-model.

Problem 2 (10+10+10+10=40)

Consider a system with a β-function

$$\beta(u) = \rho - \rho_* - (u - u_*)^2$$

Here, u is a coupling constant of the problem, and ρ is a tunable marginal parameter, with (ρ_*, u_*) some special values which we take to be given.

a) Find the fixed-point values og u with special emphasis on identifying two parameter-regimes of ρ which are qualitatively different.

b) Compute the correlation length of the system in each of these two parameter regimes.

c) Compute the correlation length exponent ν for $\rho > \rho_*$.

d) The exponent ν has an unusual feature in the context of critical phenomena. Explain what this feature is, and what gives rise to it.

Some formulae that might be useful (in all cases, it will be assumed that the candidate understands the meaning of the symbols):

1. Ginzburg-Landau theory of a superfluid

$$E = \frac{\hbar^2}{2m}|\nabla\Psi|^2 + \alpha|\Psi|^2 + u|\Psi|^4$$

2. Fourier-transform $\mathcal{F}$ and inverse transform $\mathcal{F}^{-1}$

$$\begin{aligned}
\mathcal{F}(f(\mathbf{r})) &= \tilde{f}(\mathbf{q}) = \int d^d\mathbf{r}\; e^{i\mathbf{q}\cdot\mathbf{r}}\; f(\mathbf{r}) \\
\mathcal{F}^{-1}(\tilde{f}(\mathbf{q})) &= f(\mathbf{r}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\; e^{-i\mathbf{q}\cdot\mathbf{r}}\; \tilde{f}(\mathbf{q})
\end{aligned}$$

3. Sine-integral $Si(x)$

$$\begin{aligned}
Si(x) &= \int_0^x du\, \frac{\sin u}{u} \\
\lim_{x\to\infty} Si(x) &= \frac{\pi}{2}
\end{aligned}$$

4.

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} g(q) f(\mathbf{r}\cdot\mathbf{q}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} \frac{dq}{(2\pi)^3}\; q^2\; g(q) f(qr\cos\theta)$$

5. Scaling relations for critical exponents:

$$\begin{aligned}
\alpha &= 2 - d\nu \\
\gamma &= \nu(2-\eta) \\
\alpha + 2\beta + \gamma &= 2 \\
\gamma &= \beta(\delta - 1)
\end{aligned}$$

6. Behavior of a correlation length ξ close to a critical point:

$$\xi \sim |T - T_c|^{-\nu}$$

7. Helicity modulus

$$\begin{aligned}
\Upsilon &= \left.\frac{\partial^2 F}{\partial\delta\theta^2}\right|_{\delta\theta=0} \\
Z &= \int D\theta e^{-H} = e^{-F}
\end{aligned}$$