Final exam FY 3106/FY 8303

Phase Transitions and Critical Phenomena

May 19, 2017

Problem 1

a) Isotropic ferromagnet in zero external field:

$$\Delta F_L = \frac{r}{2!}(\vec{M})^2 + \frac{u}{4!}(\vec{M}^2)^2 + \frac{w}{6!}(\vec{M}^2)^3 + \cdots$$

$$= \frac{r}{2!} M^2 + \frac{u}{4!} M^4 + \cdots$$

↑ $|\vec{M}| = M$.

Dimensionality is irrelevant for Landau theory.

Minimization:

$$\frac{\partial \Delta F_L}{\partial M} = M\left(r + \frac{u}{3!} M^2\right) = 0$$

↑ $r = a\tau$

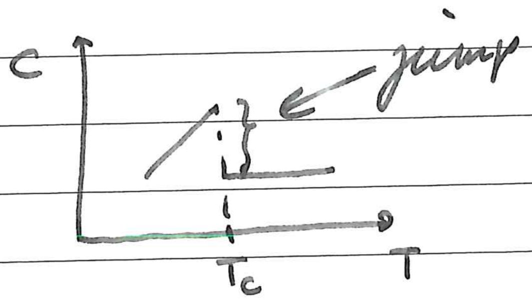
Spontaneous magnetization:

$$M_0 = \begin{cases} 0 & \tau > 0 \\ \sqrt{\frac{6a(-\tau)}{u}} & \tau < 0 \end{cases} \Rightarrow \boxed{\beta = \frac{1}{2}}$$

$$C = \left(\frac{dQ}{dT}\right)_{H=0} = -T\left(\frac{\partial^2 \Omega}{\partial T^2}\right)_{H=0}$$

where $\Omega(T,H) = \min_M \Delta F_L(T,H,M)$

$$C \sim \begin{cases} 0 & \tau > 0 \\ T_c \frac{3a^2}{u} & \tau < 0 \end{cases}$$



$\Rightarrow \boxed{\alpha = 0}$

b) $$H = \frac{1}{2}\int d\vec{x} \left\{ (\vec{\nabla} \vec{m}(\vec{x}))^2 + a\tau\, \vec{m}(\vec{x})^2 \right\}$$

Renormalization group:

$\vec{x}_\ell = \vec{x}/\ell$

$\vec{m}_\ell = c\,\vec{m}$

Coefficient in front of $(\vec{\nabla} \cdot \vec{m}(\vec{x}))^2$

term unchanged.

$$H_\ell = \frac{1}{2}\int d\vec{x}_\ell\, \ell^d \left\{ \ell^{-2} c^{-2} (\vec{\nabla}_\ell \cdot \vec{m}_\ell(\vec{x}_\ell))^2 + u\tau\, c^{-2}\, \vec{m}_\ell(\vec{x}_\ell)^2 \right\}$$

$$\ell^{d-2} c^{-2} = 1 \longrightarrow \tau_\ell = \ell^d c^{-2} \tau$$

$$= \ell^2 \tau$$

$$\Longrightarrow \boxed{y_\tau = 2}$$

c)

$$\vec{h}\cdot\int d\vec{x}\, \vec{m}(\vec{x}) \xrightarrow{RG} \vec{h}\cdot\int d\vec{x}_\ell\, \ell^d c^{-1} \vec{m}_\ell(\vec{x}_\ell)$$

$$= \ell^{d-\frac{d-2}{2}}\, \vec{h}\cdot\int d\vec{x}_\ell \cdot \vec{m}_\ell(\vec{x}_\ell)$$

$$\rightarrow \vec{h}_\ell = \ell^{\frac{d+2}{2}} \vec{h}$$

$$\Longrightarrow \boxed{y_h = \frac{d+2}{2}}$$

Hence,

$$\underline{\alpha_s = 2 - \frac{d}{y_\tau}} = \underline{\frac{4-d}{2}}, \quad \underline{\beta_s} = \frac{d-y_h}{y_\tau} = \underline{\frac{d-2}{4}}$$

d)

When $d > 4$, $\alpha_s < \alpha = 0$; $\beta_s > \beta = \frac{1}{2}$.

This means that the Landau contribution dominates over the gaussian fluctuation contribution in both C and M_0.

When $d < 4$, $\alpha_s > \alpha = 0$ and $\beta_s < \beta = \frac{1}{2}$

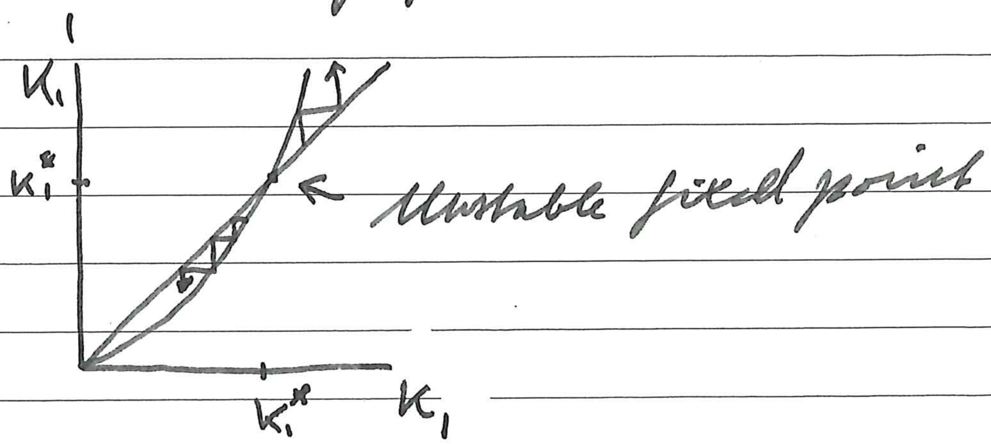
The fluctuation contribution dominates over the Landau contribution.

$\Rightarrow$ Landau ok for $d > d_c = 4$.

Problem 2

a) See derivation in Hauge, "Go critical!", pages 135-136.

b)



When $K_1 < K_1^*$, K_1 iterates to zero.
When $K_1 > K_1^*$, K_1 iterates to infinity.

$$K_1^* = 2K_1^* 4 (K_1^*)^2 \Rightarrow K_1^* = \frac{1}{4}\ln(1+2\sqrt{2}).$$

c) When $h \neq 0$, the Ising model is not critical. The critical fixed point must therefore be at $h = 0$. Hence, it remains unchanged from the $O(V)$ calculation.

Also the high temperature fixed point will be in $h = 0$ since h as all other coupling constants are reduced with increasing temperature (remember they are scaled with $-1/k_B T$).

The low temperature fixed point is changed, but this fixed point is not interesting.

For y_h, see p. 140 in Hange.

Problem 3

a) $\omega_s(\tau, h=0) \sim \tau^{2-\alpha}$

(Since $C = -T\left(\frac{\partial^2 \Omega}{\partial T^2}\right)_{H=0}$)

$$\omega_s(\tau, 0) = \frac{1}{\ell^d}\,\omega_s(\ell^{y_\tau}\tau, 0)$$

$$\tau^{2-\alpha} \sim \ell^{-d}(\ell^{y_\tau}\tau)^{2-\alpha}$$

$$\Rightarrow \quad -d + y_\tau (2-\alpha) = 0$$

$$\Rightarrow \quad \alpha = 2 - \frac{d}{y_\tau}$$

b) Correlation length:

$$\xi(\tau) \sim \tau^{-\nu}$$

$$\xi(l^{y_\tau}\tau) = \frac{1}{l}\xi(\tau) \Rightarrow$$

$$l^{-y_\tau \nu}\tau^{-\nu} = l^{-1}\tau^{-\nu}$$

$$\Rightarrow \nu = \frac{1}{y_\tau}$$

From a): $\underline{2-\alpha} = \frac{d}{y_\tau} = \underline{d\nu}$

which is the hyperscaling relation.

c) Order parameter $m \sim \tau^\beta \sim \frac{\partial}{\partial h}\omega_s(\tau,h)\Big|_{h=0}$

$$\omega_s(\tau,h) = \frac{1}{l^d}\omega_s(l^{y_\tau}\tau, l^{y_h}h)$$

$$\tau^\beta \sim l^{-d} l^{y_h}(l^{y_\tau}\tau)^\beta \Rightarrow$$

$$-d + y_h + \beta y_\tau = 0 \quad \Rightarrow$$

$$\underline{\underline{\beta = (d-y_h)/y_\tau}}$$