Exam FY3106 / FY8303
December 2021

Problem 1

a) H is dimensionless: $Z = \int \mathcal{D}\theta \, e^{-H}$

Thus: $[\rho] \cdot L^{d-2} (L^0)^2 = L^0$

$$[\rho] = L^{2-d}$$

Scaling-dimension of ρ: $\underline{2-d}$

b) $|\langle\psi\rangle|^2 = \sim |t|^{2\beta}$ $\quad (|\langle\psi\rangle| \sim |t|^\beta)$

One way of arriving at H would be to start with

$$E = \frac{\hbar^2}{2m} |\nabla\psi|^2 + \alpha |\psi|^2 + \frac{u}{2} |\psi|^4$$

with $\psi = |\psi| e^{i\theta} \simeq |\psi_0| e^{i\theta}$

Then $\rho = \frac{\hbar^2}{m} |\psi_0|^2$

$\rho \sim |\psi_0|^2$ $\sim$ density

$|\langle\psi\rangle|^2 \sim$ density

$\Upsilon \sim \rho$ (unnormalized)

Thus, if $\Upsilon \sim |t|^{\omega}$

then it is natural to compare

ω to 2β

since $\rho \sim |\Psi_0|^2$

c) $$\Upsilon \sim \rho \sim \xi^{2-d}$$

$$\sim |t|^{\nu(d-2)}$$

where we have used

$$\xi \sim |t|^{-\nu}$$

Thus $\underline{\omega = \nu(d-2)}$

Scaling relations

$$\alpha = 2 - d\nu$$

$$\alpha + 2\beta + \gamma = 2$$

$$\gamma = \nu(2-\eta)$$

From this, we obtain

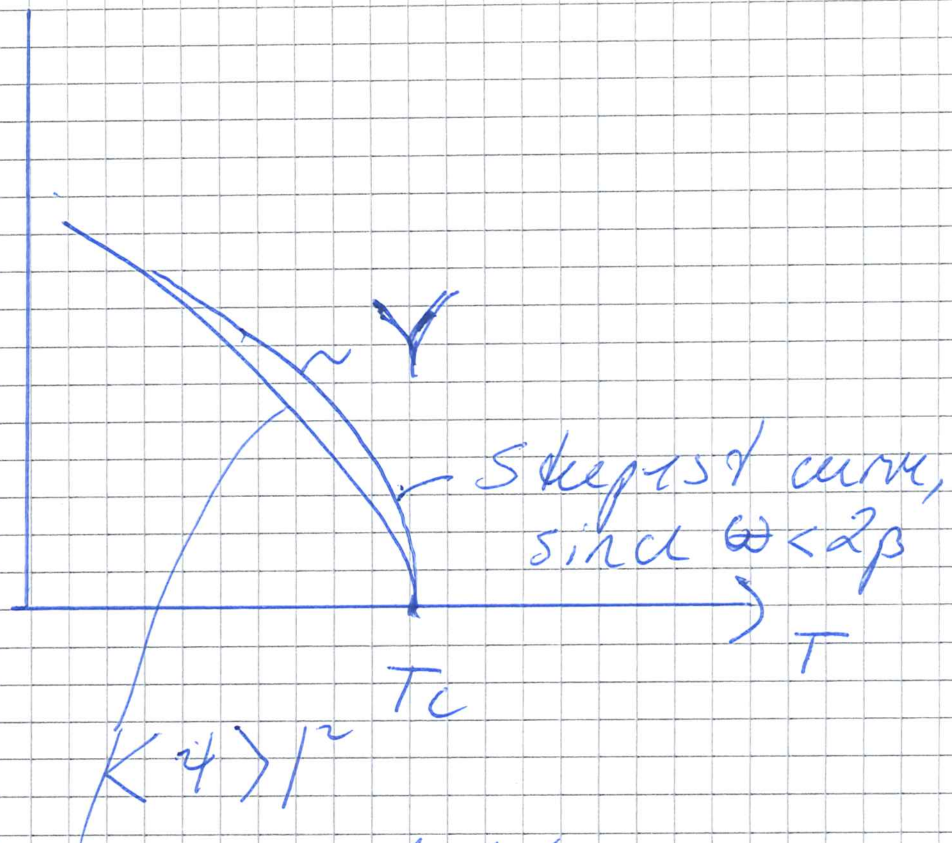
$$2 - d\nu + 2\beta + 2\nu - \eta\nu = 2$$

$$\Rightarrow \quad \nu(d-2) = 2\beta - \eta\nu$$

$$\omega = 2\beta - \eta\nu$$

If $\eta > 0$, then $\omega < 2\beta$

d)



Comment: Several candidates used

$$2\beta - \omega = 2\beta - \nu(d-2)$$

This is correct, but makes it less obvious if $2\beta > \omega$ or $2\beta < \omega$. One could have proceeded by:

(x) $2\beta - \omega = 2\beta - \nu(d-2)$

$= 2\beta - 2 + 2 - d\nu + 2\nu$

$= \alpha + 2\beta - 2 + \gamma + \eta\nu$

$= \underbrace{\alpha + 2\beta + \gamma - 2}_{=0} + \eta\nu = \underline{\eta\nu}$

(x) Just stopping here, does not employ scaling laws

Problem 2

$$\langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}\,') \rangle = \langle e^{i\theta(r)} e^{-i\theta(r')} \rangle$$

$$= e^{\underbrace{-\frac{1}{2K}\int \frac{d^d q}{(2\pi)^d}\, 2 \left[\frac{1-\cos \vec{q}\cdot(\vec{r}-\vec{r}\,')}{q^2}\right]}_{\equiv I}}$$

$$H = \frac{g}{2}\int d^d r \,(\nabla\theta)^2$$

Set $\vec{r}\,' = 0$

$\underline{d=3}$ $\qquad \vec{q}\cdot\vec{r} = qr\cos\theta$

Do the angular integrations over φ and θ (azimuthal and polar angles), and $\cos(\vec{q}\cdot\vec{r}) = \mathrm{Re}\left(e^{iqr\cos\theta}\right)$

$$I = \frac{1}{K}\,\frac{1}{(2\pi)^3}\,4\pi \int_{1/L}^{1/a} dq \left(\frac{1-\sin qr}{qr}\right)$$

a: Short-distance cutoff $= 1$

L: Long-distance cutoff

$$I = \frac{1}{K}\,\frac{2}{(2\pi)^2}\,\frac{1}{r}\int_{r/L}^{r/a} dx \left(\frac{1-\sin x}{x}\right)$$

$L \to \infty \implies$

$$I = \frac{1}{2\pi^2 K} \frac{1}{r} \left\{ \frac{r}{a} - Si\left(\frac{r}{a}\right) \right\}$$

$$= \frac{1}{2\pi^2 K} \left[1 - \frac{1}{r} Si(r) \right]$$

$Si(x)$: Sine-integral

$$Si(x) = \int_0^x du \frac{\sin u}{u} \xrightarrow{x \to \infty} \frac{\pi}{2}$$

$$\lim_{r \to \infty} I(r) = \frac{1}{2\pi^2 K}$$

$$- \frac{1}{2\pi^2 K} \left(1 - \frac{Si(r)}{r} \right)$$

$$\langle \vec{S}(\vec{r}) \cdot \vec{S}(0) \rangle = e^{-\frac{1}{2\pi^2 K}}$$

$$\xrightarrow{r \to \infty} e \qquad \neq 0$$

b)

$$\langle S(\vec{r}) \rangle = e^{-\frac{1}{4\pi^2 K}} \neq 0$$

Spin-waves do very little to reduce $T=0$ magnetization in 3d FMs.

Problem 3

a) With the constraint removed the partition function reads

$$Z = \sum_{\{\vec{b}\}} e^{-\frac{1}{2K} \sum_{\vec{r}} \vec{b}^2}$$

$$\vec{b} = (b_1, b_2, \dots b_d) \quad \text{for } d \text{ dimensions}$$

There are d b_μ's at every lattice site

$$Z = (Z_1)^{Nd} \qquad N = \#\text{ lattice sites}$$

$$Z_1 = \sum_{b=-\infty}^{\infty} e^{-\frac{1}{2K} b^2}$$

This Gaussian sum may be performed explicitly using the Jacobi θ-function

$$\vartheta_3(z,\tau) = \sum_{b=-\infty}^{\infty} q^{b^2} \eta^{b}$$

$$q = e^{\pi i \tau} \quad ; \quad \eta = e^{2\pi i z}$$

$$q = e^{-\frac{1}{2K}} = e^{i\pi\tau} \Rightarrow i\pi\tau = -\frac{1}{2K}$$

$$\tau = \frac{i}{2\pi K}$$

$$Z_1 = \vartheta_3\left(0, \frac{i}{2\pi K}\right)$$

ϑ_3 is an analytic function of its arguments, so Z_1 and hence $\ln Z_1$ and $\ln Z$ are also analytic functions of τ $\Rightarrow$ No phase-transition

b) $H = +J\sum_{\langle i,j\rangle} \left(1-\cos(\theta_i-\theta_j)\right) - h\sum_i \cos\theta_i$

Use a Villain-approximation in both terms, to write

$$Z = \int D\theta \sum_{\{\vec{m}\}} \sum_{\{Q\}} e^{-\frac{K}{2}\sum_{\vec{r},\mu}\left(\Delta_\mu\theta - 2\pi m_\mu\right)^2} \cdot e^{-\frac{\beta h}{2}\sum_{\vec{r}}\left(\theta - 2\pi Q\right)^2}$$

$K \equiv \beta J \quad ; \quad \omega \equiv \beta h$

Next, use the identity

$$\int_{-\infty}^{\infty} dy\, e^{-\frac{a}{2}y^2 + ixy} \sim e^{-\frac{x^2}{2a}}$$

twice, to obtain

$$Z = \int D\theta \sum_{\{\vec{m}\}} \sum_{\{Q\}} \int D\vec{B} \int DR \; e^{-\frac{1}{2K}\sum_{\vec{r}} \vec{B}^2 - \frac{1}{2\omega}\sum_{\vec{r}} R^2} \, e^{i\sum_{\vec{r},\mu} B_\mu\left(\Delta_\mu\theta - 2\pi m_\mu\right)} \, e^{i\sum_{\vec{r}} R\left(\theta - 2\pi Q\right)}$$

Collect terms involving θ

$$\left(\text{use } \sum_{F} \vec{B} \cdot \Delta\theta = -\sum_{F} \theta (\Delta \cdot \vec{B}) \right)$$

to obtain a factor

$$e^{i \sum_{F} \theta \left(R - \vec{\Delta} \cdot \vec{B} \right)}$$

Perform θ-integration $\Rightarrow$

at every lattice site, we have a constraint

$$\vec{\Delta} \cdot \vec{B} = R$$

Next, use Poisson summation formula

$$\sum_{\vec{n}=-\infty}^{\infty} e^{-2\pi i \vec{n} \cdot \vec{B}} = \sum_{\vec{b}=-\infty}^{\infty} \delta(\vec{b} - \vec{B})$$

$$\sum_{a=-\infty}^{\infty} e^{-2\pi i a R} = \sum_{l=-\infty}^{\infty} \delta(l - R)$$

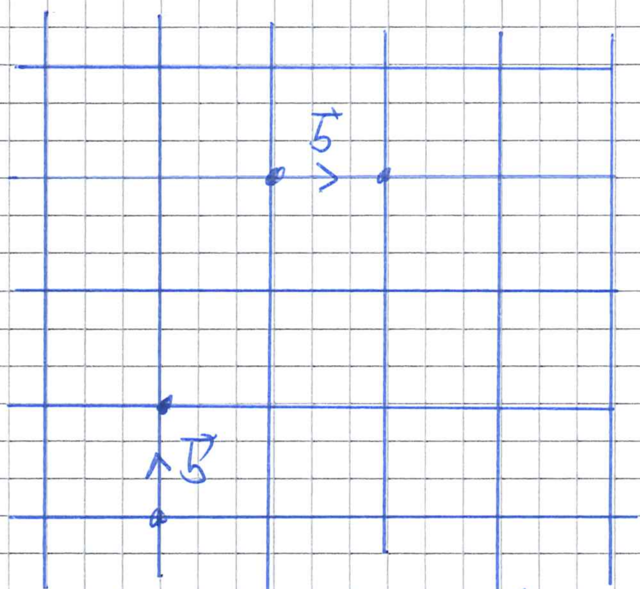
The partition function then becomes

$$Z = \sum_{\{\vec{b}\}} \sum_{\{l\}} e^{-\frac{1}{2K}\sum_{\vec{r}} \vec{b}^2 - \frac{1}{2\omega}\sum_{\vec{r}} l^2} \left(\prod_{\vec{r}} \delta_{\nabla\cdot\vec{b},\, l} \right)$$

$\omega \neq 0$ allows for $l \neq 0$

$$\nabla \cdot \vec{b} = l \neq 0$$

means that link-currents can start or end on a lattice point.



Such configurations have a non-zero probability of occurring even at very high T.

One may „glue" together such small „dumbbells" to form large loops even at very high T.

In the superfluid language, one would say one has superfluidity even at very high T.

In the spin-language, one would say the system is ordered even at high T.

Finite h destroys the

<u>order-disorder transition</u>

In fact, l is allowed to take any value (integer)

Thus $\quad \vec{\nabla}\cdot\vec{b} = l$

is ___no___ constraint on $\vec{b}$!

This destroys the PT, since

$$Z = (Z_1 Z_2)^{Nd}$$

where Z_1 and Z_2

are Jacobi θ-functions.

__c)__ $\quad h \to 0 \Rightarrow \omega \to 0$

$$\frac{1}{2\omega} \to \infty$$

This suppresses all l's except $l=0$ at each lattice site. Thus we have

$$Z = \sum_{\{\vec{b}\}} e^{-\frac{1}{2K}\sum_{r} \vec{b}^2} \left(\prod_{r} \delta_{\vec{\nabla}\cdot\vec{b},0}\right)$$

which is the loop-gas representation of the XY-model.