NTNU

Department of Physics

Examination paper for FY3107 Mathematical approximation methods in physics

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Examination time (from-to): 9-13

Permitted examination support material: C

Approved calculator

Rottmann: Matematisk formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of three problems, each containing several subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved.

Some formulas can be found on the page following the problems.

Language: English

Number of pages (including front page and attachments): 5

Informasjon om trykking av eksamensoppgave

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Date Signature

Problem 1.

Consider the ODE

$$\frac{d^2g}{ds^2} + \frac{1}{s}\frac{dg}{ds} - \left(1 + \frac{\nu^2}{s^2}\right) g = 0, \tag{1}$$

known as the modified Bessel equation. Here ν is a parameter which you may assume to be real and nonnegative.

a) Determine the location and type of the singular point(s) of (1) in the extended complex s-plane.

b) Find the indicial exponents at the point $s = 0$.

c) Show that the possible leading behaviours of solutions of (1) for s real as $s \to +\infty$ are given by $g(s) \sim C_{\pm} s^{-1/2} e^{\pm s}$, where $C_{\pm}$ are arbitrary constants.

Consider the ODE

$$\frac{d^2 f}{dx^2} - xf = 0, \tag{2}$$

known as the Airy equation.

d) Describe briefly (maximum half a page or so, and without doing any calculations) the role played by the Airy equation in the derivation of the WKB quantization condition.

The remainder of this problem concerns a relationship between solutions of the Airy equation and solutions of the modified Bessel equation for a particular value of ν.

e) For a function $f(x)$ satisfying (2), and with x real and positive, do the change of variables

$$f(x) = x^{\beta} I(s(x)), \tag{3}$$

where β is a real and positive parameter and $I(s)$ and $s(x)$ are functions (all undetermined so far). Show that (2) can then be rewritten as

$$\frac{d^2 I}{ds^2} + \left(\frac{2\beta}{xs'} + \frac{s''}{(s')^2}\right)\frac{dI}{ds} + \left(\frac{\beta(\beta-1)}{x^2(s')^2} - \frac{x}{(s')^2}\right) I = 0, \tag{4}$$

where a prime denotes differentiation with respect to x.

f) Let $s(x) = Dx^{\gamma}$ where γ and D are both real and positive parameters. Determine the unique values of γ, D, β and ν such that (4) takes the form of the modified Bessel equation (1).

Problem 2.

Consider the ODE

$$y' + y = \frac{1}{x}. \tag{5}$$

a) Use the method of dominant balance to determine the leading behaviour of $y(x)$ as $x \to +\infty$.

b) Find the full asymptotic expansion of $y(x)$ as $x \to +\infty$. (Here you should not make use of the exact solution of (5).)

c) Estimate the error in using the "optimal asymptotic approximation" to extract a numerical value for $y(x)$ from its asymptotic expansion, for some large value of x. (If you haven't found the asymptotic expansion in (b), describe how you would estimate the error for a general asymptotic expansion.)

d) Let the initial condition for (5) be $y(a) = A$ for some $a > 0$. Show that the exact solution of this initial-value problem can be written

$$y(x) = Ae^{a-x} + e^{-x} \int_a^x dt \, \frac{e^t}{t}. \tag{6}$$

e) Find the full asymptotic expansion of $y(x)$ $(x \to +\infty)$ from the exact solution (6).

Problem 3.

Consider the boundary-value problem

$$\epsilon y'' + a(x)y' + b(x)y = 0, \qquad y(0) = A, \quad y(1) = B, \tag{7}$$

with $\epsilon \to 0^+$.

a) State the conditions on the functions $a(x)$ and $b(x)$ for the solution $y(x)$ to have one boundary layer, of thickness ϵ, located at (i) $x = 0$, (ii) $x = 1$.

Next, we specialize from (7) to the boundary-value problem

$$\epsilon y'' + (\alpha + x^2)y' - y = 0, \qquad y(0) = y(1) = 1, \tag{8}$$

where α is a constant.

First, take $\alpha = 1$.

b) The solution has one boundary layer. What is its location and thickness?

c) Find the outer solution ($y_{\text{outer}}(x)$).

d) Find the inner solution ($y_{\text{inner}}(x)$).

e) Find the uniform approximation ($y_{\text{uniform}}(x)$).

Finally, take $\alpha = 0$ in (8).

f) The solution has one boundary layer. Determine its location and thickness.

Formulas

Some formulas that may be helpful:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad (n \to \infty)$$

$$\int^x dt\, \frac{1}{1+t^2} = \arctan x$$