Department of Physics

Examination paper for FY3107/FY8304 Mathematical Approximation Methods in Physics

Examination date: 5 December 2020

Examination time (from-to): 09.00-13.00 (work on exam problems)
+ 13.00-13.30 (digitization & file upload)

Permitted examination support material: A / All support material is allowed

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OTHER INFORMATION

This exam consists of four problems, each containing subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved. Under normal circumstances, all subproblems will be given the same or approximately the same weight in the marking.

Make your own assumptions: If a question is unclear/vague, make your own assumptions and specify them in your answer. Only contact academic contact in case of errors or insufficiencies in the question set.

Saving: Answers written in Inspera Assessment are automatically saved every 15 seconds. If you are working in another program remember to save your answer regularly.

Cheating/Plagiarism: The exam is an individual, independent work. Examination aids are permitted, but make sure you follow any instructions regarding citations. During the exam it is not permitted to communicate with others about the exam questions, or distribute drafts for solutions. Such communication is regarded as cheating. All submitted answers will be subject to plagiarism control. *Read more about cheating and plagiarism here.*

Citations: Any external sources used should be cited according to NTNU guidelines on *Using and citing sources*.

Notifications: If there is a need to send a message to the candidates during the exam (e.g. if there is an error in the question set), this will be done by sending a notification in Inspera. A dialogue box will appear. You can re-read the notification by clicking the bell icon in the top right-hand corner of the screen. All candidates will also receive an SMS to ensure that nobody misses out on important information. Please keep your phone available during the exam.

ABOUT SUBMISSION

File upload: All files must be uploaded before the examination time expires. 30 minutes are added to the examination time to manage the sketches/calculations/files. (The additional time is included in the remaining examination time shown in the top left-hand corner.)

How to digitize your sketches/calculations
How to create PDF documents
Remove personal information from the file(s) you want to upload

NB! You are responsible to ensure that you upload the correct file(s) for all questions. Check the file(s) you have uploaded by clicking “Download” when viewing the question. All files can be removed or replaced as long as the test is open.

The additional 30 minutes are reserved for submission. If you experience technical problems during upload/submission, you must contact technical support before the examination time expires. If you can’t get through immediately, hold the line until you get an answer.

Your answer will be submitted automatically when the examination time expires and the test closes, if you have answered at least one question. This will happen even if you do not click “Submit and return to dashboard” on the last page of the question set. You can reopen and edit your answer as long as the test is open. If no questions are answered by the time the examination time expires, your answer will not be submitted.

Withdrawing from the exam: If you become ill, or wish to submit a blank test/withdraw from the exam for another reason, go to the menu in the top right-hand corner and click “Submit blank”. This cannot be undone, even if the test is still open.

Accessing your answer post-submission: You will find your answer in Archive when the examination time has expired.

Problem 1.

Consider the ODE

$$y'' + \frac{1}{4x}y' - \frac{1}{x^3}y = 0. \tag{1}$$

a) Find and classify the singular points of (1) in the extended complex plane.

b) Determine the possible leading behaviours of $y(x)$ as $x \to 0^+$.

c) Find the full asymptotic expansion for the solution that goes to 0 as $x \to 0^+$.

d) Determine the possible leading behaviours of $y(x)$ as $x \to +\infty$.

Problem 2.

a) Determine the leading behaviour as $x \to +\infty$ of

$$\int_0^2 dt\, \cos[x(t^2 - 2t)] \tag{2}$$

b) Determine the leading behaviour as $x \to +\infty$ of

$$\int_0^\infty dt\, e^{-xt-1/t^2} \tag{3}$$

c) Show that

$$\int_0^\infty dt\, e^{-x\sinh(2t)} \sim \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(2k+1)\Gamma(k+1/2)}{\Gamma(k+1)\Gamma(1/2)}(-1)^k x^{-2k-1} \qquad (x \to +\infty) \tag{4}$$

(The following *may* be helpful: $\Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a)$ and $\Gamma(1/2) = \sqrt{\pi}$.)

d) For large x, estimate the number of terms in the optimal asymptotic approximation for the asymptotic series in (4).

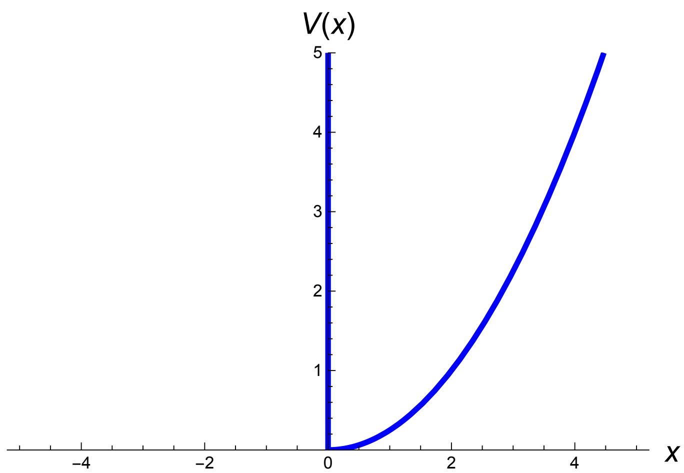
Problem 3.

In this problem we consider a Schrödinger-type equation

$$\epsilon^2 y'' = Q(x)y. \tag{5}$$

In the lectures we combined the WKB method in the physical-optics approximation with asymptotic matching to derive an approximation to $y(x)$ satisfying the boundary condition $y(+\infty) = 0$, for the case that $Q(x)$ has one turning point. The turning point was assumed to be simple, i.e. $Q(x)$ vanishes *linearly* at the turning point, with a positive and finite slope.

a) In suitably scaled variables the wavefunction $y(x)$ of a particle with energy E quantized due to confinement to a potential well $V(x)$ satisfies (5) with $Q(x) = V(x) - E$.



Assume that the potential $V(x)$ takes the form

$$V(x) = \begin{cases} \infty & x \leq 0 \\ f(x) & x > 0 \end{cases} \tag{6}$$

where the function $f(x)$ has a positive slope for $x > 0$. A typical example for $V(x)$ is shown in the figure. By making use of the known approximate solution for $y(x)$ for the problem with one simple turning point described above, derive a condition for the quantized eigenvalues E_n $(n = 0, 1, 2, \ldots)$.

b) Use the eigenvalue condition derived in (a) to predict the eigenvalues E_n when $f(x)$ corresponds to the potential of a 1-dimensional harmonic oscillator of mass m and angular frequency ω.

Problem 4.

Consider the initial-value problem

$$y'' + y + \epsilon y^2 y' = 0, \qquad y(0) = 0,\ y'(0) = 1. \tag{7}$$

Assume an expansion for $y(t)$ on the form

$$y(t) \sim Y_0(t,\tau) + \epsilon Y_1(t,\tau) + \dots, \qquad (\epsilon \to 0^+), \tag{8}$$

where $\tau = \epsilon t$.

a) Show that $Y_0(t,\tau)$ can be written on the form $Y_0(t,\tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}$, and find the equation that $A(\tau)$ should satisfy in order to avoid problematic terms in the analysis.

b) Determine $Y_0(t,\tau)$.