Solutions to eksamination in
FY8304/FY3107 Mathematical approximation methods in physics
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1a) A set of equations of the form

$$\dot{x}_j(t) = f_j(x_1(t), x_2(t), \ldots, x_n(t)) \;, \qquad j = 1, 2, \ldots, n \;,$$

is autonomous when the functions f_j have no explicit dependence on t, only an implicit dependence on t through the t-dependent variables $x_j(t)$.

A non-autonomous set of equations, of the form

$$\dot{x}_j(t) = f_j(x_1(t), x_2(t), \ldots, x_n(t), t) \;, \qquad j = 1, 2, \ldots, n \;,$$

can be made autonomous by the inclusion of the extra equation

$$\dot{t} = 1 \;. \tag{1}$$

1b) Hamilton's equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \;, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} \;.$$

are autonomous if the Hamiltonian H does not depend explicitly on t, that is, if

$$H = H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) \;.$$

Then the time derivative of H is

$$\dot{H} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j}\,\dot{q}_j + \frac{\partial H}{\partial p_j}\,\dot{p}_j \right) = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q_j} \right) = 0 \;.$$

In other words, H is a constant of motion.

1c) Write the equations as $\dot{x} = f_x(x,y)$, $\dot{y} = f_y(x,y)$. There are two fixed points, $(x,y) = (1,0)$ and $(x,y) = (-1,0)$. To determine their stability we look at the eigenvalues of the derivative matrix

$$M = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x & 0 \end{pmatrix} .$$

The trace and determinant of M are $\tau = \mathrm{Tr}\, M = 0$, $\Delta = \det M = -2x$.

At the fixed point $(x,y) = (1,0)$ we have that

$$M = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} ,$$

and $\tau = 0$, $\Delta = -2$. An eigenvalue λ is a root of the characteristic equation

$$\det(M - \lambda I) = \lambda^2 - \tau\lambda + \Delta = \lambda^2 - 2 = 0 \;.$$

The eigenvalues are $\lambda_\pm = \pm\sqrt{2}$, and the corresponding eigenvectors are

$$V_\pm = \begin{pmatrix} 1 \\ \pm\sqrt{2} \end{pmatrix} .$$

The linearized equations of motion close to the fixed point are

$$\begin{pmatrix} \dot\epsilon_x \\ \dot\epsilon_y \end{pmatrix} = M \begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} ,$$

valid for $x = 1 + \epsilon_x$, $y = \epsilon_y$, where the ϵ's are small (infinitesimal) deviations. The general solution is

$$\begin{pmatrix} \epsilon_x(t) \\ \epsilon_y(t) \end{pmatrix} = c_+ \mathrm{e}^{\lambda_+ t}\, V_+ + c_- \mathrm{e}^{\lambda_- t}\, V_- = c_+ \mathrm{e}^{\sqrt{2}\,t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_- \mathrm{e}^{-\sqrt{2}\,t} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} ,$$

with arbitrary real coefficients $c_\pm$. The fixed point is a saddle point, since it has an unstable direction V_+ and a stable direction V_-.

At the fixed point $(x, y) = (-1, 0)$ we have that

$$M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} ,$$

and $\tau = 0$, $\Delta = 2$. The characteristic equation is

$$\lambda^2 - \tau\lambda + \Delta = \lambda^2 + 2 = 0 .$$

The eigenvalues are $\lambda_\pm = \pm\mathrm{i}\sqrt{2}$, and the corresponding eigenvectors are

$$V_\pm = \begin{pmatrix} 1 \\ \pm\mathrm{i}\sqrt{2} \end{pmatrix} .$$

The fixed point is marginally stable, since the eigenvalues are purely imaginary. The linearized equations of motion close to the fixed point have the general solution

$$\begin{pmatrix} \epsilon_x(t) \\ \epsilon_y(t) \end{pmatrix} = c\, \mathrm{e}^{\lambda_+ t}\, V_+ + c^* \mathrm{e}^{\lambda_- t}\, V_- = 2\,\mathrm{Re} \left\{ c\, \mathrm{e}^{\mathrm{i}\sqrt{2}\,t} \begin{pmatrix} 1 \\ \mathrm{i}\sqrt{2} \end{pmatrix} \right\} ,$$

where c is now an arbitrary complex coefficient, $c = a + \mathrm{i}b$ with a and b real. Thus

$$\begin{aligned} \begin{pmatrix} \epsilon_x(t) \\ \epsilon_y(t) \end{pmatrix} &= 2\,\mathrm{Re} \left\{ (a + \mathrm{i}b)\, (\cos(\sqrt{2}\,t) + \mathrm{i}\sin(\sqrt{2}\,t)) \begin{pmatrix} 1 \\ \mathrm{i}\sqrt{2} \end{pmatrix} \right\} \\ &= 2a \begin{pmatrix} \cos(\sqrt{2}\,t) \\ -\sqrt{2}\, \sin(\sqrt{2}\,t) \end{pmatrix} - 2b \begin{pmatrix} \sin(\sqrt{2}\,t) \\ \sqrt{2}\, \cos(\sqrt{2}\,t) \end{pmatrix} . \end{aligned}$$

The motion is in the clockwise direction. The fixed point looks like a centre, since the orbits of the linearized equations of motion are periodic. In order to confirm that it really is a centre, with periodic solutions of the full nonlinear equations of motion, we look for a constant of motion.

We note that the equations of motion are of Hamiltonian form,

$$\dot{x} = y = \frac{\partial H}{\partial y} \ , \qquad \dot{y} = x^2 - 1 = -\frac{\partial H}{\partial x} \ ,$$

with Hamiltonian

$$H = \frac{y^2}{2} - \frac{x^3}{3} + x \ .$$

Hence H is a constant of motion, and the exact orbits are level curves of H. The fixed point $(-1,0)$ is a local minimum of H, since the first order derivatives

$$\frac{\partial H}{\partial x} = -x^2 + 1 \ , \qquad \frac{\partial H}{\partial y} = y$$

vanish there, and the second derivative matrix (the Hessian)

$$\begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial y \partial x} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -2x & 0 \\ 0 & 1 \end{pmatrix}$$

is positive definite when $x < 0$. This implies that the orbits around the fixed point $(-1,0)$ are closed, and it completes the proof that this fixed point is a centre.

1d) At the fixed point $(1,0)$ the value of the Hamiltonian is $H = 2/3$. The equation

$$H = \frac{y^2}{2} - \frac{x^3}{3} + x = \frac{2}{3}$$

defines the homoclinic orbit. It starts out from the fixed point at time $\to -\infty$ in the unstable direction $-V_+$, asymptotically as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \underset{t \to -\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathrm{e}^{\sqrt{2}\,(t-t_1)}\, V_+ = \begin{pmatrix} 1 - \mathrm{e}^{\sqrt{2}\,(t-t_1)} \\ -\sqrt{2}\,\mathrm{e}^{\sqrt{2}\,(t-t_1)} \end{pmatrix} ,$$

where t_1 is some constant time. It goes once around the other fixed point $(-1,0)$, and returns to the same fixed point $(1,0)$ at time $t \to +\infty$ in the stable direction V_-, asymptotically as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \underset{t \to +\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathrm{e}^{-\sqrt{2}\,(t-t_2)}\, V_- = \begin{pmatrix} 1 - \mathrm{e}^{-\sqrt{2}\,(t-t_2)} \\ \sqrt{2}\,\mathrm{e}^{-\sqrt{2}\,(t-t_2)} \end{pmatrix} ,$$

where t_2 is some other constant time.

1e)

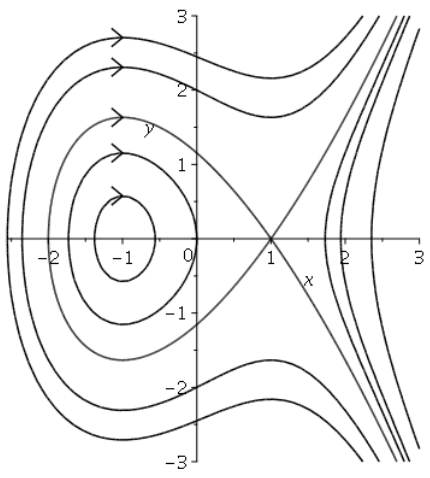

Figure 1: Phase portrait. Shows that the fixed point $(-1, 0)$ is a centre, whereas $(1, 0)$ is a saddle point, with a homoclinic orbit leaving it at $t = -\infty$ and returning at $t = +\infty$.

1f) A fixed singularity of a solution of a differential equation is at a value of the independent variable (t in the present case) where some coefficient in the equation is singular.

The set of equations $\dot{x} = y$, $\dot{y} = x^2 - 1$ has no fixed singularity.

A spontaneous (movable) singularity is a singularity of the solution at a value of t where the equation is not singular.

Now assume that $\dot{x} = y$, $\dot{y} = x^2 - 1$, and that we start at some $t = t_0$ with

$$x(t_0) = x_0 > 1 , \qquad y(t_0) = y_0 > 0 .$$

Then we know that

$$\dot{x}(t) > y_0 , \qquad \dot{y}(t) > x_0^2 - 1$$

for all $t > t_0$. Hence we conclude that $x(t) \to +\infty$ and $y(t) \to +\infty$ as t increases, either in the limit $t \to +\infty$ or perhaps already at some finite value of t.

Next, we use the fact that the Hamiltonian $H(x, y)$ has a constant value $H_0 = H(x_0, y_0)$. It follows that

$$y = \sqrt{\frac{2x^3}{3} - 2x + 2H_0} .$$

For x sufficiently large we have for example that

$$\dot{x} = y > \frac{x^{3/2}}{2} .$$

Hence we get a lower limit for $x(t)$ by integrating the equation

$$\dot{x} = \frac{x^{3/2}}{2} ,$$

which we rewrite as

$$\frac{\mathrm{d}x}{x^{3/2}} = \frac{\mathrm{d}t}{2} \,.$$

The general solution is

$$-\frac{2}{\sqrt{x}} = \frac{t - t_3}{2} \,,$$

or equivalently,

$$x(t) = \frac{16}{(t - t_3)^2} \,,$$

where t_3 is an integration constant. Remember that this was a lower limit to the exact solution, which must therefore blow up at some $t < t_3$.

This proves that every solution entering the region $x > 1$, $y > 0$ has a spontaneous singularity.

2a) Differentiating Cauchy's integral formula n times we get that

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \oint_C \mathrm{d}t \, \frac{f(t)}{(t - z)^{n+1}} \,.$$

Take $f(z) = \mathrm{e}^z$ and $z = 0$, then we get that

$$1 = \frac{n!}{2\pi\mathrm{i}} \oint_C \mathrm{d}t \, \frac{\mathrm{e}^t}{t^{n+1}} \,.$$

2b) To find the saddle point s_0 we solve the equation

$$g'(s) = (n+1) \left(\frac{\mathrm{e}^s}{s}\right)^n \left(\frac{\mathrm{e}^s}{s} - \frac{\mathrm{e}^s}{s^2}\right) = (n+1)\, g(s) \left(1 - \frac{1}{s}\right) = 0 \,.$$

The solution is $s = s_0 = 1$.

The second derivative is

$$g''(s) = (n+1) \left[g'(s) \left(1 - \frac{1}{s}\right) + \frac{g(s)}{s^2}\right] = (n+1)\, g(s) \left[(n+1) \left(1 - \frac{1}{s}\right)^2 + \frac{1}{s^2}\right],$$

hence

$$g''(1) = (n+1)\, g(1) = (n+1)\, \mathrm{e}^{n+1} > 0 \,.$$

If we write $s = u + \mathrm{i}v$ with u and v real, then

$$g''(s) = \frac{\mathrm{d}^2 g}{\mathrm{d}s^2} = \frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 g}{\partial (\mathrm{i}v)^2} = -\frac{\partial^2 g}{\partial v^2} \,.$$

It follows that

$$\left.\frac{\partial^2 g}{\partial u^2}\right|_{s=1} = -\left.\frac{\partial^2 g}{\partial v^2}\right|_{s=1} = g''(1) = (n+1)\, \mathrm{e}^{n+1} > 0 \,.$$

When we go along the real axis in the complex s plane, $g(s)$ is real and has a minimum at $s = 1$. At $s = 1$ we may go in a direction perpendicular to the real axis, then $g(s)$ remains real to first and second order in $s-1$, and has a maximum at $s = 1$, with a second derivative going to $-\infty$ as $n \to +\infty$. According to the method of steepest descent, we should take the the curve C to go through $s = 1$, perpendicular to the real axis. The main contribution to the integral comes from a small part of the curve close to $s = 1$. Hence we write $s = 1 + \mathrm{i}v$, then we introduce a small $\epsilon > 0$ and write

$$\frac{1}{n!} \sim \frac{1}{2\pi\mathrm{i}\,(n+1)^n} \int_{-\epsilon}^{\epsilon} \mathrm{i}\,\mathrm{d}v\; g(1+\mathrm{i}v) = \frac{1}{2\pi\,(n+1)^n} \int_{-\epsilon}^{\epsilon} \mathrm{d}v\; \frac{\mathrm{e}^{n+1}\,\mathrm{e}^{\mathrm{i}\,(n+1)\,v}}{(1+\mathrm{i}v)^{n+1}}\;.$$

Here v is small, and the given formula

$$1 + \mathrm{i}v = \mathrm{e}^{\mathrm{i}v - \frac{(\mathrm{i}v)^2}{2} + \dots} = \mathrm{e}^{\mathrm{i}v + \frac{v^2}{2} + \dots}$$

becomes useful. We get that

$$\frac{1}{n!} \sim \frac{\mathrm{e}^{n+1}}{2\pi\,(n+1)^n} \int_{-\epsilon}^{\epsilon} \mathrm{d}v\; \mathrm{e}^{-\frac{(n+1)v^2}{2}} \sim \frac{\mathrm{e}^{n+1}}{2\pi\,(n+1)^n} \int_{-\infty}^{\infty} \mathrm{d}v\; \mathrm{e}^{-\frac{(n+1)v^2}{2}}\;.$$

Changing integration variable to $w = v\,\sqrt{(n+1)/2}$ we get that

$$\frac{1}{n!} \sim \frac{\mathrm{e}^{n+1}}{\sqrt{2(n+1)}\;\pi\;(n+1)^n} \int_{-\infty}^{\infty} \mathrm{d}w\; \mathrm{e}^{-w^2} = \frac{\mathrm{e}^{n+1}}{\sqrt{2\pi}\,(n+1)^{n+\frac{1}{2}}}\;.$$

This is Stirling's formula.

3a) The singularities at finite x are where $\sin x = 0$, that is, $x = n\pi$ for $n = 0, \pm 1, \pm 2, \dots$, and where $\cos x = 0$, that is, $x = (n + \frac{1}{2})\pi$ for $n = 0, \pm 1, \pm 2, \dots$.

There must be a very bad singularity at infinity, since there are infinitely many singularities in any neighbourhood of infinity. So we forget about infinity and consider only the singularities at finite x.

These are all regular singular points, since the singularities of the coefficients $1/\sin x$ and $1/\cos x$ are just simple poles.

3b) Consider the singularity at $x = n\pi$. Write $x = n\pi + \xi$ where ξ is small. Then

$$\sin x = \sin(n\pi)\cos\xi + \cos(n\pi)\sin\xi = (-1)^n \sin\xi = (-1)^n \xi \left(1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} + \dots\right),$$

and

$$\begin{aligned}\frac{1}{\sin x} &= (-1)^n\,\frac{1}{\xi}\left(1 + \frac{\xi^2}{6} - \frac{\xi^4}{120} + \dots + \left(\frac{\xi^2}{6} - \frac{\xi^4}{120} + \dots\right)^2 + \dots\right) \\ &= (-1)^n\,\frac{1}{\xi}\left(1 + \frac{\xi^2}{6} + \frac{7\xi^4}{360} + \dots\right).\end{aligned}$$

Also

$$\cos x = \cos(n\pi)\cos\xi - \sin(n\pi)\sin\xi = (-1)^n \cos\xi = (-1)^n \left(1 - \frac{\xi^2}{2} + \dots\right),$$

and

$$\frac{1}{\cos x} = (-1)^n \left(1 + \frac{\xi^2}{2} + \ldots\right) .$$

Trying a power series solution

$$y(x) = \sum_k a_k \xi^k$$

we get the equation

$$\sum_k a_k \left[k(k-1)\xi^{k-2} + (-1)^n k \left(\xi^{k-2} + \frac{\xi^k}{6} + \frac{7\xi^{k+2}}{360} + \ldots\right) \right.$$
$$\left. + (-1)^n \left(\xi^k + \frac{\xi^{k+2}}{2} + \ldots\right)\right] = 0 .$$

The sum over k should be over $k = \alpha, \alpha+2, \alpha+4, \alpha+6, \ldots$ for some α such that $a_\alpha \neq 0$. In order to satisfy this equation in the limit $\xi \to 0$ we must require that

$$a_\alpha \, \alpha[\alpha - 1 + (-1)^n] \, \xi^{\alpha-2} = 0 .$$

Thus α must satisfy the indicial equation

$$\alpha[\alpha - 1 + (-1)^n] = 0 .$$

We have to distinguish between the two cases n even or n odd.

Assume first that n is even. Then the equation is

$$\sum_k a_k \left[k^2 \xi^{k-2} + k\left(\frac{\xi^k}{6} + \frac{7\xi^{k+2}}{360} + \ldots\right) + \left(\xi^k + \frac{\xi^{k+2}}{2} + \ldots\right)\right] = 0 , \qquad (2)$$

and the indicial equation is $\alpha^2 = 0$ with the unique solution $\alpha = 0$. Hence, in equation (2) we should sum over $k = 0, 2, 4, 6, \ldots$. The equation, written explicitly, is then

$$4a_2 + a_0 + \left(16a_4 + \frac{4a_2}{3} + \frac{a_0}{2}\right)\xi^2 + \ldots = 0 .$$

The terms shown vanish when we take a_0 arbitrary and

$$a_2 = -\frac{a_0}{4} , \qquad a_4 = -\frac{a_2}{12} - \frac{a_0}{32} .$$

Further recursion relations determine successively a_k for $k = 6, 8, 10, \ldots$.

Unfortunately, this procedure gives only one solution, whereas a second order equation must have two linearly independent solutions. We know what the second solution should look like, it should have the form

$$y(x) = y_2(x) + y_1(x) \, \ln \xi$$

with

$$y_1(x) = \sum_{k=0,2,4,\ldots} a_k \xi^k , \qquad y_2(x) = \sum_{k=0,2,4,\ldots} b_k \xi^k .$$

We have then that

$$
\begin{aligned}
y'(x) &= y_2'(x) + \frac{y_1(x)}{\xi} + y_1'(x)\,\ln\xi\,, \\
y''(x) &= y_2''(x) + \frac{2y_1'(x)}{\xi} - \frac{y_1(x)}{\xi^2} + y_1''(x)\,\ln\xi\,.
\end{aligned}
$$

Define the differential operator

$$
L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\sin x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{\cos x}\,.
$$

Then

$$
Ly = Ly_2 + \frac{2}{\xi}\,y_1' + \left(-\frac{1}{\xi^2} + \frac{1}{\xi\sin x}\right) y_1 + (Ly_1)\ln\xi\,.
$$

In order to get $Ly = 0$ we should require that y_1 satisfies the homogeneous equation $Ly_1 = 0$, and that y_2 satisfies the inhomogeneous equation

$$
Ly_2 = -\frac{2}{\xi}\,y_1' + \left(\frac{1}{\xi^2} - \frac{1}{\xi\sin x}\right) y_1 = -\frac{2}{\xi}\,y_1' - \left(\frac{1}{6} + \frac{7\xi^2}{360} + \ldots\right) y_1\,,
$$

where the dots represent terms of order ξ^4, ξ^6, and so on. The power series expansions of $y_1(x)$ and $y_2(x)$ give first equation (2) for the coefficients a_k, and then the equation

$$
\begin{aligned}
\sum_{k=0,2,4,\ldots} b_k \left[k^2\xi^{k-2} + k\left(\frac{\xi^k}{6} + \frac{7\xi^{k+2}}{360} + \ldots\right) + \left(\xi^k + \frac{\xi^{k+2}}{2} + \ldots\right)\right] \\
= \sum_{k=0,2,4,\ldots} a_k \left[-2k\xi^{k-2} - \frac{\xi^k}{6} - \frac{7\xi^{k+2}}{360} + \ldots\right] \qquad (3)
\end{aligned}
$$

for the coefficients b_k. The last equation more explicitly written out is

$$
4b_2 + b_0 + \left(16b_4 + \frac{4b_2}{3} + \frac{b_0}{2}\right)\xi^2 + \ldots = -4a_2 - \frac{a_0}{6} + \left(-8a_4 - \frac{a_2}{6} - \frac{7a_0}{360}\right)\xi^2 + \ldots\,.
$$

To satisfy the two equations (2) and (3) we can take a_0 and b_0 arbitrary, then

$$
a_2 = -\frac{a_0}{4}\,, \qquad a_4 = -\frac{a_2}{12} - \frac{a_0}{32}\,,
$$

as before, and

$$
b_2 = -\frac{b_0}{4} - a_2 - \frac{a_0}{24}\,, \qquad b_4 = -\frac{b_2}{12} - \frac{b_0}{32} - \frac{a_4}{2} - \frac{a_2}{96} - \frac{7a_0}{5760}\,.
$$

Further recursion relations determine successively a_k and b_k for $k = 6, 8, 10, \ldots$.

Note that if we take $a_0 = 0$ and $b_0 \neq 0$, we just recover the power series solution without the logarithm. The logical choice in order to define a new solution is to take $a_0 \neq 0$ and $b_0 = 0$.

Now to the case where n is odd. Then the equation is

$$
\sum_k a_k \left[k(k-2)\xi^{k-2} - k\left(\frac{\xi^k}{6} + \frac{7\xi^{k+2}}{360} + \ldots\right) - \left(\xi^k + \frac{\xi^{k+2}}{2} + \ldots\right)\right] = 0\,, \qquad (4)
$$

and the indicial equation is $\alpha(\alpha-2)=0$ with the two solutions $\alpha=0$ and $\alpha=2$. Hence, in equation (4) we should sum over even k starting with either $k=0$ or $k=2$. The equation, written explicitly, is then

$$-a_0+\left(8a_4-\frac{4a_2}{3}-\frac{a_0}{2}\right)\xi^2+\ldots=0\,.$$

The terms shown vanish when we take $a_0=0$, a_2 arbitrary, and

$$a_4=\frac{a_2}{6}\,.$$

Further recursion relations determine successively a_k for $k=6,8,10,\ldots$.

Again we get only one solution, and we have to look for a second solution of the form

$$y(x)=y_2(x)+y_1(x)\,\ln\xi$$

with

$$y_1(x)=\sum_{k=2,4,6,\ldots} a_k\xi^k\,,\qquad y_2(x)=\sum_{k=0,2,4,\ldots} b_k\xi^k\,.$$

In order to get $Ly=0$ we should require that y_1 satisfies the homogeneous equation $Ly_1=0$ (that is why we sum from $k=2$ instead of from $k=0$), and that y_2 satisfies the inhomogeneous equation

$$Ly_2=-\frac{2}{\xi}\,y_1'+\left(\frac{1}{\xi^2}-\frac{1}{\xi\sin x}\right)y_1=-\frac{2}{\xi}\,y_1'+\left(\frac{2}{\xi^2}+\frac{1}{6}+\frac{7\xi^2}{360}+\ldots\right)y_1\,,$$

where the dots represent terms of order ξ^4,ξ^6, and so on. The power series expansions of $y_1(x)$ and $y_2(x)$ give the equation (4) for the coefficients a_k, and the equation

$$\sum_{k=0,2,4,\ldots} b_k\left[k(k-2)\xi^{k-2}-k\left(\frac{\xi^k}{6}+\frac{7\xi^{k+2}}{360}+\ldots\right)-\left(\xi^k+\frac{\xi^{k+2}}{2}+\ldots\right)\right]$$
$$=\sum_{k=2,4,6,\ldots} a_k\left[-2(k-1)\xi^{k-2}+\frac{\xi^k}{6}+\frac{7\xi^{k+2}}{360}+\ldots\right]\qquad(5)$$

for the coefficients b_k. The last equation more explicitly written out is

$$-b_0+\left(8b_4-\frac{4b_2}{3}-\frac{b_0}{2}\right)\xi^2+\ldots=-2a_2+\left(-6a_4+\frac{a_2}{6}\right)\xi^2+\ldots\,.$$

This gives that $a_0=0$, a_2 is arbitrary, and

$$a_4=\frac{a_2}{6}\,,$$

as before. Then it gives that $b_0=2a_2$, b_2 is arbitrary, and

$$b_4=\frac{b_2}{6}+\frac{b_0}{16}-\frac{3a_4}{4}+\frac{a_2}{48}\,.$$

Further recursion relations determine successively a_k and b_k for $k=6,8,10,\ldots$.

Note that instead of saying that a_2 is arbitrary and $b_0 = 2a_2$, we may turn it around and say that b_0 is arbitrary and $a_2 = b_0/2$. If we take $b_0 = 2a_2 = 0$ and $b_2 \neq 0$, we just recover the power series solution without the logarithm. Hence, the logical way of getting a new solution is to take $b_0 = 2a_2 \neq 0$, but we may take $b_2 = 0$.

So far the singularity at $x = n\pi$. Consider now the singularity at $x = (n + \frac{1}{2})\pi$. Write $x = (n + \frac{1}{2})\pi + \xi$ where ξ is small. Then

$$\begin{aligned}\sin x &= \sin\left(\left(n + \frac{1}{2}\right)\pi\right)\cos\xi + \cos\left(\left(n + \frac{1}{2}\right)\pi\right)\sin\xi \\ &= (-1)^n\cos\xi = (-1)^n\left(1 - \frac{\xi^2}{2} + \ldots\right),\end{aligned}$$

and

$$\frac{1}{\sin x} = (-1)^n\left(1 + \frac{\xi^2}{2} + \ldots\right).$$

Also

$$\begin{aligned}\cos x &= \cos\left(\left(n + \frac{1}{2}\right)\pi\right)\cos\xi - \sin\left(\left(n + \frac{1}{2}\right)\pi\right)\sin\xi \\ &= -(-1)^n\sin\xi = (-1)^{n+1}\xi\left(1 - \frac{\xi^2}{6} + \ldots\right),\end{aligned}$$

and

$$\frac{1}{\cos x} = (-1)^{n+1}\frac{1}{\xi}\left(1 + \frac{\xi^2}{6} + \ldots\right).$$

Trying a power series solution

$$y(x) = \sum_k a_k\xi^k$$

we get the equation

$$\begin{aligned}\sum_k a_k\Bigg[k(k-1)\xi^{k-2} + (-1)^n k\left(\xi^{k-1} + \frac{\xi^{k+1}}{2} + \ldots\right) \\ +(-1)^{n+1}\left(\xi^{k-1} + \frac{\xi^{k+1}}{6} + \ldots\right)\Bigg] = 0\,. \qquad (6)\end{aligned}$$

The sum over k should be over $k = \alpha, \alpha+1, \alpha+2, \alpha+3, \ldots$ for some α such that $a_\alpha \neq 0$. In order to satisfy this equation in the limit $\xi \to 0$ we must require that

$$a_\alpha\,\alpha(\alpha - 1)\,\xi^{\alpha-2} = 0\,.$$

Thus α must satisfy the indicial equation

$$\alpha(\alpha - 1) = 0\,,$$

with solutions $\alpha = 0$ and $\alpha = 1$. The equation more explicitly written out is

$$\begin{aligned}(-1)^{n+1}a_0\xi^{-1} + 2a_2 + \left(6a_3 + (-1)^n a_2 + (-1)^{n+1}\,\frac{a_0}{6}\right)\xi \\ + \left(12a_4 + (-1)^n 2a_3 + (-1)^n\,\frac{a_1}{3}\right)\xi^2 + \ldots = 0\,.\end{aligned}$$

It gives that $a_0 = a_2 = 0$, a_1 is arbitrary, and

$$\begin{aligned} a_3 &= (-1)^{n+1}\,\frac{a_2}{6} + (-1)^n\,\frac{a_0}{36} = 0\;, \\ a_4 &= (-1)^{n+1}\,\frac{a_3}{6} + (-1)^{n+1}\,\frac{a_1}{36} = (-1)^{n+1}\,\frac{a_1}{36}\;. \end{aligned}$$

Further recursion relations determine successively a_k for $k = 5, 6, 7, \ldots$.

Here again we get only one solution, and we have to consider solutions of the form

$$y(x) = y_2(x) + y_1(x)\;\ln\xi$$

with

$$y_1(x) = \sum_{k=1,2,3,\ldots} a_k\xi^k\;, \qquad y_2(x) = \sum_{k=0,1,2,3,\ldots} b_k\xi^k\;.$$

In order to get $Ly = 0$ we should require that y_1 satisfies the homogeneous equation $Ly_1 = 0$, and that y_2 satisfies the inhomogeneous equation

$$Ly_2 = -\frac{2}{\xi}\,y_1' + \left(\frac{1}{\xi^2} - \frac{1}{\xi\sin x}\right) y_1 = -\frac{2}{\xi}\,y_1' + \left(\frac{1}{\xi^2} - (-1)^n\left(\frac{1}{\xi} + \frac{\xi}{2} + \ldots\right)\right) y_1\;,$$

where the dots represent terms of order ξ^4, ξ^6, and so on. The power series expansions of $y_1(x)$ and $y_2(x)$ give the equation (6) for the coefficients a_k, and the equation

$$\begin{aligned} &\sum_{k=0}^{\infty} b_k\left[k(k-1)\xi^{k-2} + (-1)^n k\left(\xi^{k-1} + \frac{\xi^{k+1}}{2} + \ldots\right) + (-1)^{n+1}\left(\xi^{k-1} + \frac{\xi^{k+1}}{6} + \ldots\right)\right] \\ &\qquad = \sum_{k=1}^{\infty} a_k\left[-2(k-1)\xi^{k-2} + (-1)^{n+1}\left(\xi^{k-1} + \frac{\xi^{k+1}}{2} + \ldots\right)\right] \end{aligned} \tag{7}$$

for the coefficients b_k. The last equation more explicitly written out is

$$\begin{aligned} &(-1)^{n+1}b_0\xi^{-1} + 2b_2 + \left(6b_3 + (-1)^n\left(b_2 - \frac{b_0}{6}\right)\right)\xi + \ldots \\ &= -2a_2 + (-1)^{n+1}a_1 + \left(-4a_3 + (-1)^{n+1}\left(a_2 + \frac{a_0}{2}\right)\right)\xi + \ldots\;. \end{aligned}$$

This gives that $a_0 = a_2 = a_3 = 0$, a_1 is arbitrary, and

$$a_4 = (-1)^{n+1}\,\frac{a_1}{36}\;,$$

as before. Then it gives that $b_0 = 0$, b_1 is arbitrary, and

$$\begin{aligned} b_2 &= -2a_2 + (-1)^{n+1}a_1 = (-1)^{n+1}a_1\;, \\ b_3 &= -\frac{2a_3}{3} + (-1)^{n+1}\left(a_2 + \frac{a_0}{2}\right) = 0\;. \end{aligned}$$

Further recursion relations determine successively a_k and b_k for $k = 4, 5, 6, \ldots$.

In the examination only the leading asymptotic behaviour at the singular points was asked for. A sufficient answer is the following:

- For $x = n\pi + \xi$ with n an even integer and ξ small we have either $y(x) \sim 1$ or $y(x) \sim \ln \xi$.
- For $x = n\pi + \xi$ with n an odd integer and ξ small we have either $y(x) \sim \xi^2$ or $y(x) \sim 2 + \xi^2 \ln \xi$.
- For $x = (n + \frac{1}{2})\pi + \xi$ with n an even or odd integer and ξ small we have either $y(x) \sim \xi$ or $y(x) \sim \xi(1 + \ln \xi)$.