FY3107/8304 Mathematical approximation methods in physics
Solution to exam, November 2018

Remarks on weighting: All subproblems (1a, ..., 3f) were given weight 1 in the marking, except 2d (weight 0.5), 3a (weight 0.75), 3b (weight 0.25), and 3e (weight 0.5).

Problem 1

(a) For a 2nd order homogeneous linear ODE

$$\frac{d^2g}{ds^2} + p(s)\frac{dg}{ds} + q(s)g(s) = 0, \tag{1}$$

a point $s = s_0$ with s_0 finite is a singular point if $p(s)$ and/or $q(s)$ are not analytic at s_0. A singular point s_0 is a regular singular point (RSP) if both $(s - s_0)p(s)$ and $(s - s_0)^2 q(s)$ are analytic at s_0. Otherwise the singular point s_0 is an irregular singular pont (ISP).

For the modified Bessel equation, $p(s) = 1/s$ and $q(s) = -(1 + \nu^2/s^2)$. It follows that $s = 0$ is an RSP for this ODE, and that it has no other singular points in the complex plane.

The extended complex plane consists of the complex plane plus the "point at infinity" ($s = \infty$). The nature of the point $s = \infty$ is determined by introducing $t = 1/s$ and analyzing the point $t = 0$ in the standard way. We get

$$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} = -\frac{1}{s^2}\frac{d}{dt} = -t^2\frac{d}{dt}, \tag{2}$$

$$\frac{d^2}{ds^2} = \left(-t^2\frac{d}{dt}\right)\left(-t^2\frac{d}{dt}\right) = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}. \tag{3}$$

Also defining $h(t) = g(s)$, the modified Bessel equation is thus transformed into

$$t^4\frac{d^2h}{dt^2} + 2t^3\frac{dh}{dt} + t(-t^2)\frac{dh}{dt} - (1 + \nu^2 t^2)h(t) = 0, \tag{4}$$

i.e.

$$\frac{d^2h}{dt^2} + \frac{1}{t}\frac{dh}{dt} - \left(\frac{1}{t^4} + \frac{\nu^2}{t^2}\right)h = 0. \tag{5}$$

The factor $1/t^4$ multiplying h implies that $t = 0$, and thus $s = \infty$, is an ISP (of the lowest rank).

(b) As $s = 0$ is an RSP, we consider a solution around $s = 0$ that takes the form of a Frobenius series with indicial exponent α:

$$g(s) = \sum_{n=0}^{\infty} a_n s^{\alpha+n} \tag{6}$$

with $a_0 \neq 0$. Differentiating this series term by term and inserting into the ODE gives

$$\sum_n a_n(\alpha+n)(\alpha+n-1)s^{\alpha+n-2} + \sum_n a_n(\alpha+n)s^{\alpha+n-2} - \sum_n a_n s^{\alpha+n} - \nu^2\sum_n a_n s^{\alpha+n-2} = 0. \tag{7}$$

The coefficients of each power s^m must sum to 0. For the smallest exponent $m = \alpha - 2$ this gives

$$a_0[\alpha(\alpha-1) + \alpha - \nu^2] = 0. \tag{8}$$

Since $a_0 \neq 0$, this gives the indicial equation $\alpha^2 - \nu^2 = 0$, i.e. the indicial exponents are

$$\alpha = \pm\nu. \tag{9}$$

(c) We expect the general solution to have an essential singularity at $s = \infty$ since this is an ISP for the ODE. Therefore we try a solution on the form $g(s) = \exp(R(s))$ (exponential substitution). This gives

$$\frac{dg}{ds} = e^R \frac{dR}{ds}, \tag{10}$$

$$\frac{d^2g}{ds^2} = e^R \left[\frac{d^2R}{ds^2} + \left(\frac{dR}{ds}\right)^2\right]. \tag{11}$$

Inserting this into the ODE and cancelling the common nonzero factor $\exp(R)$ gives

$$R'' + (R')^2 + \frac{1}{s}R' - 1 - \frac{\nu^2}{s^2} = 0 \tag{12}$$

(from now on, a prime denotes differentiation with respect to s in this subproblem). As $s \to +\infty$, $\nu^2/s^2 \ll 1$. Let us also assume that R'' and R'/s are $\ll (R')^2$. This gives

$$(R')^2 \sim 1 \quad \Rightarrow \quad R' \sim \pm 1 \qquad (s \to +\infty) \tag{13}$$

From this it follows that $R'/s \sim \pm 1/s$, so it was consistent to neglect R'/s compared to $(R')^2$ as $s \to +\infty$. The same is true for R''. Integrating gives

$$R(s) \sim \pm s \qquad (s \to +\infty). \tag{14}$$

Next we write $R(s) = \pm s + C(s)$. Thus $R' = \pm 1 + C'$ and $R'' = C''$. Inserting into (12) gives

$$C'' + (\pm 1 + C')^2 + \frac{1}{s}(\pm 1 + C') - 1 - \frac{\nu^2}{s^2} = 0. \tag{15}$$

Expanding out and cancelling equal terms gives

$$C'' \pm 2C' + (C')^2 \pm \frac{1}{s} + \frac{C'}{s} - \frac{\nu^2}{s^2} = 0. \tag{16}$$

Here $\nu^2/s^2 \ll 1/s$ and $C'/s \ll C'$. Next, differentiating the relation $C(s) \ll s$ (which follows from (14)) gives $C' \ll 1$, so $(C')^2 \ll C'$. Differentiating one more time leads us to neglect C'' too. Thus

$$\pm 2C' \sim \mp \frac{1}{s} \quad \Rightarrow \quad C' \sim -\frac{1}{2s} \qquad (s \to +\infty) \tag{17}$$

It follows that $(C')^2 \sim 1/(4s^2)$ and $C'' \sim 1/(2s^2)$, so our approximations were consistent. Integrating (17) gives

$$C(s) \sim -\frac{1}{2}\ln s \qquad (s \to +\infty). \tag{18}$$

Next we write

$$C(s) = -\frac{1}{2}\ln s + D(s). \tag{19}$$

Thus $C' = -1/(2s) + D'$ and $C'' = 1/(2s^2) + D''$. Inserting into (16) gives

$$\frac{1}{2s^2} + D'' \pm 2\left(-\frac{1}{2s} + D'\right) + \left(-\frac{1}{2s} + D'\right)^2 \pm \frac{1}{s} + \frac{1}{s}\left(-\frac{1}{2s} + D'\right) - \frac{\nu^2}{s^2} = 0. \tag{20}$$

Expanding out and cancelling equal terms gives

$$D'' \pm 2D' + (D')^2 + \frac{1}{s^2}\left(\frac{1}{4} - \nu^2\right) = 0. \tag{21}$$

Differentiating the relation $D(s) \ll \ln s$ (which follows from (18)) gives $D' \ll 1/s$ and $D'' \ll 1/s^2$, so we may neglect D'' and $(D')^2$. This gives

$$D' \sim \mp \frac{1}{2s^2}\left(\frac{1}{4} - \nu^2\right) \qquad (s \to +\infty) \tag{22}$$

Integrating the right-hand side gives

$$\pm\frac{1}{2s}\left(\frac{1}{4}-\nu^2\right)+k\equiv E(s)+k, \tag{23}$$

where k is an integration constant.[1] Since the function $E(s)\propto 1/s\ll 1$ as $s\to+\infty$,

$$D(s)\sim k \qquad (s\to+\infty). \tag{24}$$

Summarizing, we have so far found

$$R(s)\sim \pm s-\frac{1}{2}\ln s+k \qquad (s\to+\infty) \tag{25}$$

The difference between the left-hand side and the right-hand side in this asymptotic relation is

$$D(s)-k\sim E(s) \qquad (s\to+\infty) \tag{26}$$

As this difference is $\ll 1$ ($s\to+\infty$), it is permissible to exponentiate the asymptotic relation (25). This gives the leading behaviour

$$y=\exp(R(s))\sim \exp\left(\pm s-\frac{1}{2}\ln s+k\right)=Ks^{-1/2}\exp(\pm s) \qquad (s\to+\infty) \tag{27}$$

where $K=\exp(k)$ (which obviously can be different for the $\pm$ cases).

(d) One wants to find the eigenvalues E of an eigenvalue problem whose ODE is of Schrodinger type,

$$\epsilon^2 y''=(V(x)-E)y, \tag{28}$$

with $y\to 0$ as $x\to\pm\infty$. The WKB method (in the so-called physical optics approximation) is used to find approximate solutions in the various regions with $V(x)>E$ and $V(x)<E$. However, this WKB approximation breaks down in the vicinity of the so-called turning points where $V(x)=E$. Thus near the turning points a different approach is required. This consists of linearizing the potential $V(x)$ around each turning point, and then doing a simple change of variables which turns the linearized ODE into the Airy equation. The solution of this Airy equation is then matched to the WKB solutions on both sides of the turning point. For a problem with two turning points this procedure eventually leads to a condition for the discrete eigenvalues E known as the WKB quantization condition.

(e) We have

$$\frac{df}{dx} = \beta x^{\beta-1}I+x^{\beta}\frac{ds}{dx}\frac{dI}{ds}, \tag{29}$$

$$\frac{d^2f}{dx^2} = \beta(\beta-1)x^{\beta-2}I+2\beta x^{\beta-1}\frac{ds}{dx}\frac{dI}{ds}+x^{\beta}\frac{d^2s}{dx^2}\frac{dI}{ds}+x^{\beta}\left(\frac{ds}{dx}\right)^2\frac{d^2I}{ds^2}. \tag{30}$$

Inserting this into the Airy equation $d^2f/dx^2-xf=0$, dividing the equation by the coefficient function of d^2I/ds^2, and simplifying, gives

$$\frac{d^2I}{ds^2}+\left(\frac{2\beta}{xs'}+\frac{s''}{(s')^2}\right)\frac{dI}{ds}+\left(\frac{\beta(\beta-1)}{x^2(s')^2}-\frac{x}{(s')^2}\right)I=0. \qquad \text{QED.} \tag{31}$$

(f) In order to compare (31) with the modified Bessel equation, we first need to express the coefficient functions in terms of the variable s. We have

$$s' = D\gamma x^{\gamma-1}, \tag{32}$$

$$s'' = D\gamma(\gamma-1)x^{\gamma-2}. \tag{33}$$

[1] Integration constants also appear when integrating the asymptotic expressions for R' and C' above. But in these cases such a constant was $\ll$ the antiderivatives themselves and could thus be neglected in the asymptotic expressions (14) and (18) for R and C.

Let us first consider the factors in the coefficient function of dI/ds:

$$\begin{aligned} xs' &= D\gamma x^{\gamma} = \gamma s, && (34) \\ \frac{s''}{(s')^2} &= \frac{D\gamma(\gamma-1)x^{\gamma-2}}{D^2\gamma^2 x^{2(\gamma-1)}} = \frac{\gamma-1}{D\gamma x^{\gamma}} = \frac{\gamma-1}{\gamma s}. && (35) \end{aligned}$$

Thus the coefficient function of dI/ds becomes $(2\beta+\gamma-1)/(\gamma s)$. Setting this equal to the coefficient function $1/s$ of the first-derivative term in the modified Bessel equation, γ cancels out, and we find

$$\beta = \frac{1}{2}. \tag{36}$$

Next consider the factors in the coefficient function of I. From (34) we get $x^2(s')^2 = \gamma^2 s^2$. Furthermore,

$$\frac{x}{(s')^2} = \frac{x}{D^2\gamma^2 x^{2(\gamma-1)}} = \frac{1}{D^2\gamma^2 x^{2\gamma-3}} = \frac{1}{D^2\gamma^2}\left(\frac{s}{D}\right)^{(3-2\gamma)/\gamma} \tag{37}$$

Setting the coefficient function of I equal to that of the modified Bessel equation gives

$$\frac{\beta(\beta-1)}{\gamma^2 s^2} - \frac{1}{D^2\gamma^2}\left(\frac{s}{D}\right)^{(3-2\gamma)/\gamma} = -1 - \frac{\nu^2}{s^2}. \tag{38}$$

Here, the coefficients of the terms proportional to s^{-2} must be equal, giving

$$\nu^2 = \frac{\beta(1-\beta)}{\gamma^2} = \frac{1}{4\gamma^2}. \tag{39}$$

Eq. (38) furthermore dictates that

$$\frac{1}{D^2\gamma^2}\left(\frac{s}{D}\right)^{(3-2\gamma)/\gamma} = 1, \tag{40}$$

from which it follows that $3-2\gamma = 0$ and $D^2\gamma^2 = 1$. Therefore

$$\gamma = \frac{3}{2} \quad \text{and} \quad D = \frac{2}{3}. \tag{41}$$

Finally, from (39), $\nu^2 = 1/9$, i.e.

$$\nu = \frac{1}{3}. \tag{42}$$

In summary, the Airy solution $f(x) = x^{1/2} I\left(\frac{2}{3}x^{3/2}\right)$, where $I(s)$ is a solution of the modified Bessel equation for $\nu = 1/3$.[2]

Problem 2

(a) We will use the method of dominant balance, so we try the assumption that for $x \to +\infty$ two of the three terms in the ODE balance against each other and dominate the remaining term which is thus negligible in comparison. Obviously there are three possible alternatives for which term is assumed negligible. One alternative is

$$y' \ll y \sim \frac{1}{x} \quad \Rightarrow \quad y' \sim -\frac{1}{x^2} \ll y \sim \frac{1}{x} \text{ as } x \to +\infty, \tag{45}$$

[2] Consistent with this finding, it can be shown that the standard basis $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ of solutions of the Airy equation can be expressed as ($x > 0$)

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{1}{3}x^{1/2}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) - I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right], && (43) \\ \mathrm{Bi}(x) &= \frac{1}{\sqrt{3}}x^{1/2}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) + I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right], && (44) \end{aligned}$$

where $I_{\pm 1/3}(s)$ are so-called modified Bessel functions of the first kind (the functions $I_{\pm\nu}(s)$ form a basis of solutions for the modified Bessel equation when ν is not an integer).

which is consistent. The two other alternatives are not consistent as they both give outcomes that contradict the starting assumption:

$$y \ll y' \sim \frac{1}{x} \qquad \Rightarrow \qquad y \sim \ln x \gg \frac{1}{x} \text{ as } x \to +\infty \tag{46}$$

$$\frac{1}{x} \ll y' \sim -y \qquad \Rightarrow \qquad y \sim Ce^{-x} \ll \frac{1}{x} \text{ as } x \to +\infty \tag{47}$$

Thus the leading behaviour is

$$y \sim \frac{1}{x} \qquad (x \to +\infty). \tag{48}$$

(b) We factor out the leading behaviour by writing

$$y(x) = \frac{1}{x}w(x). \tag{49}$$

This gives $y' = -w/x^2 + w'/x$. Inserting these expressions into the ODE for y, we find the following ODE for w:

$$w' + \left(1 - \frac{1}{x}\right) w = 1. \tag{50}$$

We assume an asymptotic expansion of power series form for $w(x)$, i.e.

$$w(x) \sim \sum_{n=0}^{\infty} a_n x^{-\alpha n} \qquad (x \to +\infty). \tag{51}$$

As this should be an asymptotic power series for $x \to +\infty$, we must have $\alpha > 0$, and to get the correct leading behaviour for $y(x)$, we must have $a_0 = 1$. Inserting this series into (50) gives

$$\sum_n a_n(-\alpha n)x^{-\alpha n-1} + \sum_n a_n x^{-\alpha n} - \sum_n a_n x^{-\alpha n-1} \sim 1 \qquad (x \to +\infty) \tag{52}$$

We find the coefficients by comparing them for each power x^m. The largest possible exponent m is 0, which gives $a_0 = 1$ (which we already knew). Removing this term from both sides, we can rearrange to get

$$\sum_{n=1}^{\infty} a_n x^{-\alpha n} \sim \sum_{n=0}^{\infty} a_n(\alpha n + 1)x^{-\alpha n-1} \tag{53}$$

The least negative exponent on the right-hand side is $m = -1$ (occurs for $n = 0$), with coefficient $a_0 = 1$. Thus an identical term $1x^{-1}$ must appear on the left-hand side. The only[3] possibility is that this comes from the term with $n = 1$, which requires $\alpha = 1$. Inserting $\alpha = 1$ in (53) and equating coefficients of identical powers then gives

$$a_{n+1} = (n+1)a_n, \qquad n = 0, 1, 2, \ldots, \tag{54}$$

so (also using $a_0 = 1$)

$$a_n = n! \tag{55}$$

Therefore the asymptotic expansion of $y(x)$ is

$$y(x) \sim \frac{1}{x}\sum_{n=0}^{\infty} n!\, x^{-n} = \sum_{n=0}^{\infty} n!\, x^{-(n+1)} \qquad (x \to +\infty) \tag{56}$$

(c) Let me write $y(x) \sim \sum_{n=0}^{\infty} f_n(x)$ with $f_n(x) = n!\, x^{-(n+1)}$. The ratio of successive terms is therefore

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{(n+1)!x^{-(n+2)}}{n!x^{-(n+1)}} = \frac{n+1}{x}. \tag{57}$$

[3] The correct power $x^{-\alpha n} = x^{-1}$ could also be obtained by taking $n = N$ for some integer $N > 1$, which requires $\alpha = 1/N$. However, this would imply that the left-hand side also contains terms proportional to $x^{-n/N}$ for $1 \leq n < N$, which do not appear on the right-hand side. Therefore $\alpha = 1$ is the only possibility.

Thus $f_n(x)$ will first decrease with n before increasing without limit. The "optimal asymptotic approximation" consists of summing all terms up to but not including the smallest term, which is an estimate of the error. The smallest term occurs for the smallest n such that $f_{n+1}(x)/f_n(x) > 1$. Using (57) gives $n > x - 1$. Thus n is the integer between $x-1$ and x. Here I assumed that x is a generic real number, i.e. not an integer.[4] Ignoring the minute distinction between the integer n and (generically) noninteger x, we set $n = x$ in the estimate for the error:

$$f_x(x) = x!\, x^{-(x+1)} \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x x^{-(x+1)} = \sqrt{\frac{2\pi}{x}}e^{-x} \qquad (x \to +\infty) \tag{58}$$

i.e. the error decreases exponentially with x. Here I used the asymptotic (Stirling) approximation for the factorial function given in the formula set.

(d) As the ODE is 1st order inhomogeneous linear, the exact solution can be obtained using the integrating factor method. Thus we multiply the ODE by $\exp(\int^x 1dt) = \exp(x)$, giving

$$e^x y' + e^x y = \frac{d}{dx}(e^x y) = \frac{e^x}{x}. \tag{59}$$

Next, rename the independent variable as t and integrate from $t = a$ to $t = x$:

$$\int_a^x dt\, \frac{d}{dt}(e^t y(t)) = e^t y(t)\Big|_a^x = e^x y(x) - e^a y(a) = \int_a^x dt\, \frac{e^t}{t}. \tag{60}$$

Inserting $y(a) = A$ and solving for $y(x)$ gives

$$y(x) = Ae^{a-x} + e^{-x}\int_a^x dt\, \frac{e^t}{t}. \tag{61}$$

(e) The integral in (61) can not be evaluated in closed form in terms of elementary functions. To develop an asymptotic expansion, we try integration by parts: $\int vu' = vu\Big| - \int v'u$. Writing $e^t = (d/dt)e^t \equiv u'$ and $v = 1/t$ gives

$$\int_a^x dt\, \frac{e^t}{t} = \int_a^x dt\, \frac{1}{t}\frac{d}{dt}e^t = \frac{e^t}{t}\Big|_a^x - \int_a^x dt\,(-1)\frac{1}{t^2}e^t = \frac{e^x}{x} - \frac{e^a}{a} + \int_a^x dt\, \frac{e^t}{t^2}. \tag{62}$$

Now the new integral on the right can be evaluated in the same way. Iterating this process will give a series involving inverse powers of x. It will be convenient to define

$$I_n(x) \equiv e^{-x}\int_a^x dt\, \frac{e^t}{t^n} \qquad (n = 1, 2, \ldots) \tag{63}$$

This gives

$$e^x I_n(x) = \int_a^x dt\, \frac{1}{t^n}\frac{d}{dt}e^t = \frac{e^t}{t^n}\Big|_a^x - \int_a^x dt\,(-n)\frac{1}{t^{n+1}}e^t = \frac{e^x}{x^n} - \frac{e^a}{a^n} + ne^x I_{n+1}(x), \tag{64}$$

i.e.

$$I_n(x) = \frac{1}{x^n} - \frac{e^{a-x}}{a^n} + nI_{n+1}(x). \tag{65}$$

Starting with $I_1(x)$ which appears in (61), iterating a few times gives

$$\begin{aligned} I_1(x) &= \frac{1}{x} - \frac{e^{a-x}}{a} + 1\cdot I_2(x) = \frac{1}{x} + \frac{1}{x^2} - e^{a-x}\left(\frac{1}{a} + \frac{1}{a^2}\right) + 2I_3(x) \\ &= \frac{1}{x} + \frac{1}{x^2} + \frac{1\cdot 2}{x^3} - e^{a-x}\left(\frac{1}{a} + \frac{1}{a^2} + \frac{1\cdot 2}{a^3}\right) + 3I_4(x) \\ &= \frac{1}{x} + \frac{1}{x^2} + \frac{1\cdot 2}{x^3} + \frac{1\cdot 2\cdot 3}{x^4} - e^{a-x}\left(\frac{1}{a} + \frac{1}{a^2} + \frac{1\cdot 2}{a^3} + \frac{1\cdot 2\cdot 3}{a^4}\right) + 4I_5(x). \end{aligned} \tag{66}$$

[4] If x is an integer, $f_{n+1}(x)/f_n(x) = 1$ for $n = x - 1$, i.e. there are *two* smallest terms, for $n = x - 1$ and $n = x$.

The pattern is clear:

$$I_1(x) = \sum_{n=0}^{N-1} n!\, x^{-(n+1)} - e^{a-x} \sum_{m=0}^{N-1} m!\, a^{-(m+1)} + N I_{N+1}(x) \tag{67}$$

for any integer $N \geq 1$. Note that as $x \to +\infty$, any term $e^{a-x} m!/a^{-(m+1)}$ in the second sum is, due to the exponential decay in x, subdominant (i.e. it goes to 0 faster than any inverse power of x), i.e. it doesn't contribute to the asymptotic expansion of $I_1(x)$, which is thus given by the $N \to \infty$ limit of the first sum alone in (67). For the same reason, the term Ae^{a-x} in (61) does not contribute to the asymptotic expansion of $y(x)$. Thus we get

$$y(x) \sim \sum_{n=0}^{\infty} n!\, x^{-(n+1)} \qquad (x \to +\infty) \tag{68}$$

in agreement with the result in (b).

Problem 3

(a) The function $a(x)$ should satisfy $a(x) \neq 0$ for $0 \leq x \leq 1$. If $a(x) > 0$, the boundary layer will be at $x = 0$; if $a(x) < 0$, the boundary layer will be at $x = 1$. (These are the most important conditions. In addition, $a(x)$ and $b(x)$ should be continuous functions.)

(b) This problem is of the type considered in (a) with $a(x) = 1 + x^2$ and $b(x) = -1$. As $a(x) > 0$, it follows that there is a boundary layer of thickness ϵ at $x = 0$.

(c) The outer solution obeys a simplified ODE obtained by neglecting the term $\epsilon y''$:

$$(1 + x^2)y' - y = 0 \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{1 + x^2}. \tag{69}$$

Integrating gives

$$\ln y = \arctan x + \tilde{C} \quad \Rightarrow \quad y = C \exp(\arctan x). \tag{70}$$

Imposing the boundary condition at $x = 1$ gives

$$y(1) = 1 = C \exp(\arctan 1) = C \exp(\pi/4) \quad \Rightarrow \quad C = \exp(-\pi/4). \tag{71}$$

Thus

$$y(x) = \exp(\arctan x - \pi/4) \equiv y_{\text{outer}}(x). \tag{72}$$

(d) The inner solution also obeys a simplified ODE. To identify this we introduce the inner variable $X = x/\delta$ where δ is the thickness of the boundary layer. Thus $d/dx = (1/\delta) d/dX$ and $d^2/dx^2 = (1/\delta^2) d^2/dX^2$. We also introduce $Y(X) = y(x)$. Since we are considering the inner region, we may also replace $a(x)$ by its leading-order approximation $a(0) = 1$. This gives the ODE

$$\frac{\epsilon}{\delta^2} \frac{d^2Y}{dX^2} + \frac{1}{\delta} \frac{dY}{dX} - Y = 0. \tag{73}$$

From (b) we know[5] that $\delta = \epsilon$. Thus terms 1 and 2 are $O(1/\epsilon)$ while term 3 is $O(1)$. Therefore term 3 is negligible compared to terms 1 and 2 as $\epsilon \to 0^+$, giving the simplified ODE

$$\frac{d^2Y}{dX^2} + \frac{dY}{dX} = 0. \tag{74}$$

This gives $\frac{dY}{dX} = -C_2 \exp(-X)$ where C_2 is an integration constant. Integrating again gives

$$Y(X) = C_1 + C_2 \exp(-X) \quad \text{i.e.} \quad y(x) = C_1 + C_2 \exp(-x/\epsilon) \tag{75}$$

[5] Alternatively, if one doesn't know this, it can be deduced from a dominant-balance analysis, as discussed in 3(f) for a different ODE.

where C_1 is another integration constant. Imposing the boundary condition at $x = 0$ gives

$$y(0) = 1 = C_1 + C_2 \quad \Rightarrow \quad C_1 = 1 - C_2. \tag{76}$$

Thus

$$y(x) = 1 + C_2[\exp(-x/\epsilon) - 1] \equiv y_{\text{inner}}(x). \tag{77}$$

To determine C_2 one needs to match $y_{\text{inner}}(x)$ and $y_{\text{outer}}(x)$ in the overlap region[6] characterized by $x \to 0$, $X = x/\epsilon \to \infty$, as $\epsilon \to 0^+$ (these conditions are e.g. satisfied for $x = O(\sqrt{\epsilon})$), i.e. we set

$$y_{\text{outer}}(x = 0) = Y(X \to \infty) \equiv y_{\text{match}}. \tag{78}$$

This gives

$$y_{\text{match}} = \exp(-\pi/4) = 1 - C_2 \quad \Rightarrow \quad C_2 = 1 - \exp(-\pi/4). \tag{79}$$

Thus

$$y_{\text{inner}}(x) = [1 - \exp(-\pi/4)] \exp(-x/\epsilon) + \exp(-\pi/4). \tag{80}$$

(e) The uniform approximation is given by

$$\begin{aligned} y_{\text{uniform}}(x) &= y_{\text{inner}}(x) + y_{\text{outer}}(x) - y_{\text{match}} \\ &= [1 - \exp(-\pi/4)] \exp(-x/\epsilon) + \exp(\arctan x - \pi/4). \end{aligned} \tag{81}$$

(f) Now $a(x) = x^2$. The fact that $a(x) > 0$ for $x > 0$ prohibits a boundary layer at $x = 1$ or at an interior point. Thus the only possibility is a boundary layer at $x = 0$. However, since $a(x) = 0$ there, the conditions for a thickness-ϵ boundary layer are not satisfied. To deduce the thickness δ of the boundary layer, we proceed like we did at the beginning of (d) by introducing $X = x/\delta$ and $Y(X) = y(x)$. Also using $a(x) = \delta^2 X^2$ gives the ODE

$$\frac{\epsilon}{\delta^2} \frac{d^2 Y}{dX^2} + \delta X^2 \frac{dY}{dX} - Y = 0. \tag{82}$$

δ can be determined by a dominant-balance analysis: one assumes that as $\epsilon \to 0^+$ one term is negligible compared to the two other terms which balance each other. There are three alternatives:

- Term 1 is negligible: Balancing terms 2 and 3 gives $\delta = 1$. Thus terms 2 and 3 are $O(1)$, while term 1 is $O(\epsilon)$ which is indeed negligible compared to terms 2 and 3. So the initial assumption does not lead to a contradiction in itself. However, since δ is found to be independent of ϵ, this case does not describe a solution with a boundary layer (the thickness of a boundary layer should go to zero as $\epsilon \to 0$). This could have been anticipated, as term 1 is the one we would have ignored if we wanted to find the outer solution.

- Term 2 is negligible: Balancing terms 1 and 3 gives $\epsilon/\delta^2 = 1$, i.e. $\delta = \epsilon^{1/2}$. Thus terms 1 and 3 are $O(1)$ while term 2 is $O(\epsilon^{1/2})$ which is indeed negligible compared to terms 1 and 3.

- Term 3 is negligible: Balancing terms 1 and 2 gives $\epsilon/\delta^2 = \delta$, i.e. $\delta = \epsilon^{1/3}$. Thus terms 1 and 2 are $O(\epsilon^{1/3})$, while term 3 is $O(1)$ and therefore dominant, which contradicts the initial assumption.

We conclude that the boundary layer at $x = 0$ has thickness $\delta = \epsilon^{1/2}$.

[6] Since finding C_2 involves a comparison with $y_{\text{outer}}(x)$, it could be argued that the expression (77) should be considered the end of the calculation of $y_{\text{inner}}(x)$. If so, the calculation of C_2 would be done as a part of (e) instead.