# NTNU Trondheim, Institutt for fysikk

**Examination for FY3403 Particle Physics**
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Allowed tools: mathematical tables, pocket calculator
Some formulas and data can be found on p. 2ff.

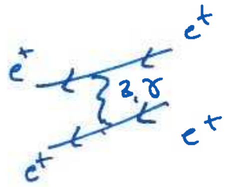
## 1. Interactions.

Which interaction (weak, strong, electromagnetic) is responsible for the following decays or scatterings and why:

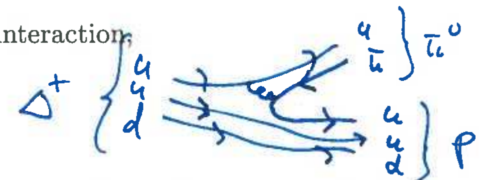
a.) $\rho^0 \to \pi^+\pi^-$ (2 pts)
b.) $e^+e^+ \to e^+e^+$ (2 pts)
c.) $\Delta^+ \to p\pi^0$ (2 pts)

a.) strong,

b) electromagnetic (plus neutral current),

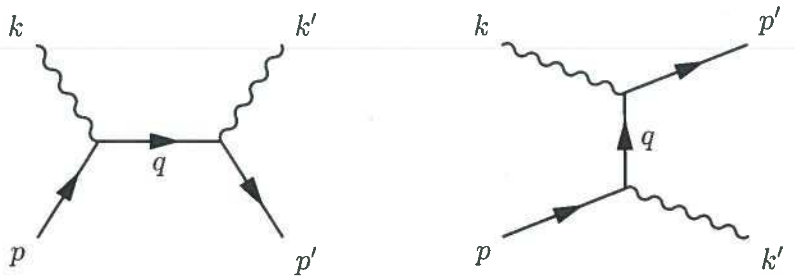


c.) strong interaction,



## 2. Feynman diagrams and amplitudes.

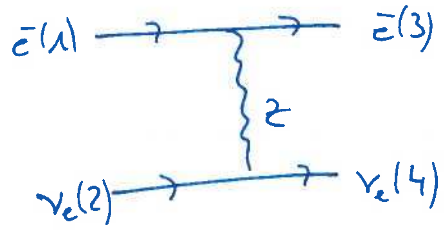
Assume that fermions are not mixed, i.e that $U_{\rm CKM} = U_{\rm PMNS} = 1$. Draw all tree-level diagrams (i.e. containing no loops) and write down the Feynman amplitude in momentum space as in the example given. If the process is not allowed, give the reason. (10 pts)
Example:

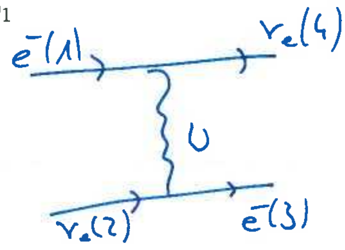



$$\mathrm{i}\mathcal{M} = -\mathrm{i}e^2\bar{u}(p')\left[\not{\varepsilon}^{*\prime}\,\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\,\not{\varepsilon} + \not{\varepsilon}\,\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\,\not{\varepsilon}^{*\prime}\right]u(p).$$

a.) $e^- + \nu_e \to e^- + \nu_e$:

With $\mathrm{i}\mathcal{M} = \mathrm{i}\mathcal{M}_Z + \mathrm{i}\mathcal{M}_W$, $k = p_3 - p_1$ and $q = p_4 - p_1$



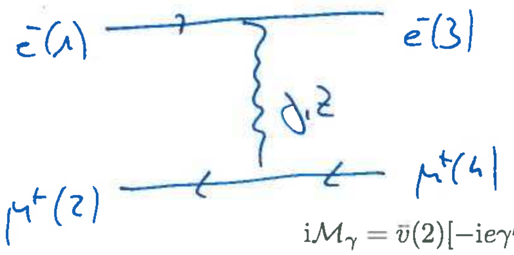




$$\mathrm{i}\mathcal{M}_Z = \bar{u}(4)[-\mathrm{i}\frac{g_Z}{2}\gamma^\mu(c_V - c_A\gamma^5)]u(2)\mathrm{i}\frac{-\eta_{\mu\nu} + k_\mu k_\nu/M_Z^2}{k^2 - M_Z^2}\bar{u}(3)[-\mathrm{i}\frac{g_Z}{2}\gamma^\mu(c_V - c_A\gamma^5)]u(1)$$

$$\mathrm{i}\mathcal{M}_W = \bar{u}(4)[-\frac{\mathrm{i}g}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)]u(1)\mathrm{i}\frac{-\eta_{\mu\nu} + q_\mu q_\nu/M_W^2}{q^2 - M_W^2}\bar{u}(3)[-\frac{\mathrm{i}g}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)]u(2)$$

b.) $e^- + \mu^+ \to e^- + \mu^+$

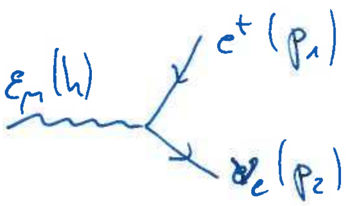
With $\mathrm{i}\mathcal{M} = \mathrm{i}\mathcal{M}_\gamma + \mathrm{i}\mathcal{M}_Z$ and $q = p_3 - p_1$




$$\mathrm{i}\mathcal{M}_\gamma = \bar{v}(2)[-\mathrm{i}e\gamma^\mu]v(4)\mathrm{i}\frac{-\eta_{\mu\nu}}{k^2}\bar{u}(3)[-\mathrm{i}e\gamma^\mu]u(1)$$

$$\mathrm{i}\mathcal{M}_Z = \bar{v}(2)[-\mathrm{i}\frac{g_Z}{2}\gamma^\mu(c_V - c_A\gamma^5)]v(4)\mathrm{i}\frac{-\eta_{\mu\nu} + k_\mu k_\nu/M_Z^2}{k^2 - M_Z^2}\bar{u}(3)[-\mathrm{i}\frac{g_Z}{2}\gamma^\mu(c_V - c_A\gamma^5)]u(1)$$

c.) $W^+ \to e^+ + \nu_e$: With $k = p_1 + p_2$




$$\mathrm{i}\mathcal{M} = \varepsilon_\mu \bar{u}(2)[-\frac{\mathrm{i}g}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)]v(1)$$

**3. Charged kaon decay.**

Consider the decay $K^- \to l^- + \bar{\nu}_l$ with $l = \{e, \mu, \tau\}$ in the limit $m_K \ll m_W$.

a.) Parametrise the unknown coupling between a $W_\mu^\pm$ boson and the kaon by the form factor $F_K^\mu$ and write down the Feynman amplitude $\mathcal{M}$ for this decay. (4 pts)

b.) Express the form factor $F_K^\mu$ by the scalar function $f_K$ ("the kaon decay constant") and

the relevant tensor(s). (3 pts)
c.) Calculate $|\bar{\mathcal{M}}|^2$ averaging over initial and summing over final spins. You should obtain (10 pts)

$$|\bar{\mathcal{M}}|^2 = |\bar{\mathcal{M}}|^2 = \left(\frac{g}{2m_W}\right)^4 f_K^2 m_l^2 (m_K^2 - m_l^2).$$

d.) Calculate the decay rate $\Gamma$. (10 pts)
e.) The ratio of the decay rates is

$$\frac{\Gamma(K^- \to \mu^- + \bar{\nu}_\mu)}{\Gamma(K^- \to e^- + \bar{\nu}_e)} = \frac{m_\mu^2(m_K^2 - m_\mu^2)^2}{m_e^2(m_K^2 - m_e^2)^2}.$$

Explain the origin of the factors. (4 pts)
f.) Explain qualitatively the effect of neutrino oscillations, if the neutrino is observed at the distance $L$ from the decay point. (4 pts)

This process corresponds to charged pion decay discussed in the lectures.
a.) In the limit $m_K \ll m_W$, we can replace

$$\mathrm{i}\frac{-\eta^{\mu\nu} + k^\mu k^\nu/m_W^2}{k^2 - m_W^2} \to \frac{\mathrm{i}\eta^{\mu\nu}}{m_W^2}.$$

Including a factor $g/(2\sqrt{2})$ twice, it follows

$$\mathcal{M} = \frac{g^2}{8m_W^2}[\bar{u}(3)\gamma_\mu(1-\gamma^5)v(2)]F_K^\mu.$$

[Any expression with $\tilde{F}_K^\mu = AF_K^\mu$, where $A$ is a number, is equally valid.]

b.) The only tensors in the game are $\eta_{\mu\nu}$ and the momentum $p^\mu$ of the kaon. We need a tensor of rank one, and thus $F_K^\mu = f_K p^\mu$.

c.) With

$$\mathcal{M}^* = \frac{g^2}{8m_W^2}[\bar{v}(2)\gamma_\nu(1-\gamma^5)u(3)]F_K^\mu.$$

it follows

$$|\bar{\mathcal{M}}|^2 = \frac{g^4 f_K^2}{64 m_W^4}\sum_{s_2,s_3}[\bar{u}(3)\gamma_\mu(1-\gamma^5)v(2)][\bar{v}(2)\gamma_\nu(1-\gamma^5)u(3)]p^\mu p^\nu.$$

Inserting the completeness relations gives

$$|\bar{\mathcal{M}}|^2 = \frac{g^4 f_K^2}{64 m_W^4}\mathrm{tr}[(\not{p}_3 - m_l)\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)]p^\mu p^\nu.$$

Using the the trace theorems, it is

$$|\bar{\mathcal{M}}|^2 = \frac{1}{8}\frac{g^4 f_K^2}{64 m_W^4}[2(pp_2)(pp_3) - p^2(p_2p_3)].$$

With $p = p_2 + p_3$, or

$$pp_2 = p_2p_3, \qquad pp_3 = m_l^2 + p_2p_3$$

and thus

$$2p_2p_3 = m_K^2 - m_l^2$$

we find

$$|\bar{\mathcal{M}}|^2 = \left(\frac{g}{2m_W}\right)^4 f_K^2 m_l^2 (m_K^2 - m_l^2).$$

Combining (13) and (14), it follows

$$\mathrm{d}\Gamma = \frac{1}{32\pi^2 m_K^2} |\mathcal{M}_{fi}|^2 |\boldsymbol{p}_{\mathrm{cms}}| \, \mathrm{d}\Omega ,$$

The angular integration gives $4\pi$, and adding

$$|\boldsymbol{p}_{\mathrm{cms}}| = \frac{1}{2m_K}(m_K^2 - m_l^2)$$

gives

$$\Gamma = \frac{1}{16\pi m_K^3} \left(\frac{g}{2m_W}\right)^4 f_K^2 m_l^2 (m_K^2 - m_l^2)^2.$$

e.) The total angular momentum of the initial state is zero. Hence the charged lepton and the neutrino have opposite spin and thus equal helicities. The weak current couple to left-chiral particles and right-chiral antiparticles, and thus the charged lepton has to flip its chirality, leading to a factor $m_l$ in the amplitude. The factor $(m_K^2 - m_l^2)$ takes into account the phase space.

f.) The neutrino state $\nu_\mu$ created in kaon decay is a superposition of mass eigenstates $\nu_i$,

$$\nu_\mu(0) = U_{\mu i}\nu_i = U_{\mu 1}\nu_1(0) + U_{\mu 2}\nu_2(0) + U_{\mu 3}\nu_3(0).$$

The phase of the mass eigenstates will evolve as

$$\nu_i(x) = \nu_i(0)\exp(\mathrm{i}p_i x) \simeq \nu_i(0)\mathrm{e}^{\mathrm{i}\phi}\exp[-\mathrm{i}Ex/(2m_i^2)]$$

The last term implies that the states devlop an oscillating phase difference as they propagate. At detection, we have to split $\nu_i(L)$ into flavor states, $\nu_i(L) = U_{i\alpha}^* \nu_\alpha$. Then $|\langle\nu_\alpha|\,\nu_i(L)\rangle|^2$ gives the probability to observe flavour alpha. As a result, the originally close to 100% pure $\nu_\mu$ state becomes a superposition of all three flavors.

## 4. The Higgs doublet.

Consider the complex scalar SU(2) doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + \mathrm{i}\phi_2 \\ \phi_3 + \mathrm{i}\phi_4 \end{pmatrix} . \qquad (1)$$

with the Lagrangian

$$\mathscr{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2. \tag{2}$$

a) Show that the Lagrangian is invariant under global SU(2) and U(1) transformations. (6 pts)
b.) Write down the corresponding Lagrangian which is invariant under local SU(2) and U(1) transformations. (6 pts)
c.) Explain the difference between the two cases $\mu^2 > 0$ and $\mu^2 < 0$. (4 pts)

a.) The scalar doublet $\Phi$ transforms as

$$\Phi \to U\phi = \exp\left\{\frac{\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right\}\Phi \qquad \textit{and} \qquad \Phi \to V\phi = \exp\{\mathrm{i}\vartheta Y/2\}\Phi,$$

while $\Phi^\dagger$ transform as

$$\Phi^\dagger \to \Phi^\dagger \exp\left\{\frac{-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right\} \qquad \textit{and} \qquad \Phi^\dagger \to \Phi^\dagger \exp\{-\mathrm{i}\vartheta Y/2\}.$$

Since for global transformation $\alpha$ and $\vartheta$ are constant and $U^\dagger U = 1$, $V^\dagger V = 1$, the transformation cancels in $\mathscr{L}$.

b.) The Lagrangian is invariant, if if one substitutes normal with covariant derivatives, $\partial^\mu \to D^\mu$.

$$\partial^\mu \Phi \to D^\mu \Phi = \left(\partial^\mu + \frac{\mathrm{i}g}{2}\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu + \frac{\mathrm{i}g'}{2}B^\mu\right)\Phi\,,$$

where the $W^\mu$ and $B^\mu$ are the gauge fields of the SU(2) and U(1) group, respectively.

c.) Sketch of $V$, case without ($\mu^2 < 0$, $v = 0$) and with SSB ($\mu$>00,, $v \neq 0$):

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Useful formulas and Feynman rules

The gamma matrices form a Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{3}$$

The matrix

$$\gamma^5 \equiv \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \,. \tag{4}$$

satisfies $(\gamma^5)^\dagger = \gamma^5$, $(\gamma^5)^2 = 1$, and $\{\gamma^\mu, \gamma^5\} = 0$.

$$\overline{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0 \,. \tag{5}$$

The trace of an odd number of $\gamma^\mu$ matrices vanishes, as well as

$$\mathrm{tr}[\gamma^5] = \mathrm{tr}[\gamma^\mu\gamma^5] = \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^5] = 0 \,. \tag{6}$$

Non-zero traces are

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\,] = 4\eta^{\mu\nu} \quad \textit{and} \quad \mathrm{tr}[\not{a}\not{b}] = 4a\cdot b \tag{7}$$

$$\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(a\cdot b)\,(c\cdot d) - (a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)] \tag{8}$$

$$\mathrm{tr}\left[\gamma^5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\right] = 4\mathrm{i}\varepsilon_{\mu\nu\alpha\beta} \tag{9}$$

Useful are also $\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b$, $\not{a}\not{a} = a^2$ and the following contractions,

$$\gamma^\mu\gamma_\mu = 4\,, \qquad \gamma^\mu\not{a}\gamma_\mu = -2\not{a}\,, \qquad \gamma^\mu\not{a}\not{b}\gamma_\mu = 4a\cdot b\,, \qquad \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}\,. \tag{10}$$

Completeness relations

$$\sum_s u_a(p,s)\bar{u}_b(p,s) = (\not{p} + m)_{ab} \,, \tag{11}$$

$$\sum_s v_a(p,s)\bar{v}_b(p,s) = (\not{p} - m)_{ab} \,. \tag{12}$$

Decay rate

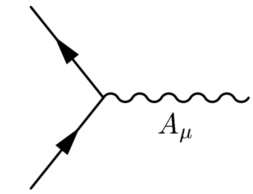
$$\mathrm{d}\Gamma_{fi} = \frac{1}{2E_i}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)} \,. \tag{13}$$

The two particle phase space $\mathrm{d}\Phi^{(2)}$ in the rest frame of the decaying particle is

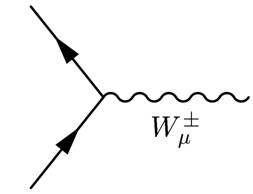
$$\mathrm{d}\Phi^{(2)} = \frac{1}{16\pi^2}\,\frac{|\boldsymbol{p}'_{\mathrm{cms}}|}{M}\,\mathrm{d}\Omega \,, \tag{14}$$

pseudoscalar mesons: $\pi^\pm$, $m = 139.6\,\mathrm{MeV}$, $(u\bar{d})$, $(d\bar{u})$.
$K^\pm$, $m = 493.68\,\mathrm{MeV}$, $(u\bar{s})$, $(s\bar{u})$.
vector mesons: $\rho^0$, $m = 775.5\,\mathrm{MeV}$, $(u\bar{u} - d\bar{d})/\sqrt{2}$.
baryons: $\Delta^+$, $m = 1232\,\mathrm{MeV}$, $(uud)$.

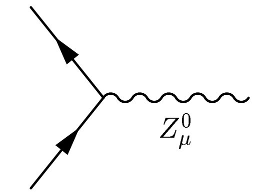
| | mass | energy | 1/length | 1/time | temperature |
|---|---|---|---|---|---|
| GeV | $1.78\times10^{-24}$ g | $1.60\times10^{-3}$ erg | $5.06\times10^{13}$ cm$^{-1}$ | $1.52\times10^{24}$ s$^{-1}$ | $1.16\times10^{13}$K |



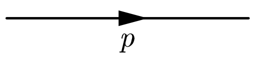

$$-\mathrm{i}e\gamma^\mu$$



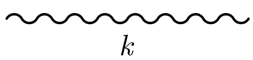

$$-\frac{\mathrm{i}g}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$



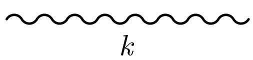

$$-\mathrm{i}\frac{g_Z}{2}\gamma^\mu(c_V-c_A\gamma^5)$$




$$\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2}$$




$$\mathrm{i}\frac{-\eta_{\mu\nu}}{k^2}$$




$$\mathrm{i}\frac{-\eta_{\mu\nu}+k_\mu k_\nu/M^2}{k^2-M^2}$$