PROBLEM 1 (60%)

(a) Draw the general primitive vertices used in Feynman diagrams for the following cases: electromagnetic interactions, charged weak interactions, neutral weak interactions, and strong interactions. In each case, explain your notation.

(b) Explain the meaning of quark generations and Cabibbo-rotated quark states and how these two concepts are related.

(c) Explain in detail, using equations as much as possible, how gauge-invariance creates a coupling between a charged spin-1/2 fermion field and a photon.

(d) Explain the concept of neutrino oscillations and how they are related to the solar neutrino problem.

Consider now a different problem. Describe what is meant by the concepts below. It is important that you present the concepts below in a clear and logical way, in effect not just by listing several 'keywords' that are relevant in each case.

(e) The helicity of a particle.

(f) The Higgs mechanism, in particular the difference between the Higgs field and the Higgs boson. You are not required to perform detailed analytical calculations here: a clear qualitative explanation will suffice.

(g) The TCP-theorem and its main consequence.

(h) The difference between colorless and color singlet particles.

(i) Renormalization in the context of Feynman diagrams.

PROBLEM 2 (40%)

Consider the decay of a muon $\mu \to e + \nu_\mu + \bar{\nu}_e$.

(a) Which interaction is responsible for this decay? Write down the lowest order Feynman diagram.

(b) Write down the Feynman amplitude $\mathcal{M}$ for this process, briefly mentioning which rules you have used to obtain it. You may assume that the momentum transfer via the gauge boson is small.

(c) Assume that the spin-states of the particles are unknown and use Casimir's trick to derive an expression for $\langle |\mathcal{M}|^2 \rangle$. This includes evaluation of any traces in your expression. *Hint:* the final answer should be in the form:

$$\langle |\mathcal{M}|^2 \rangle = K F(p_1, p_2) G(p_3, p_4), \tag{1}$$

where K is a constant and $F = F(p_1, p_2)$ and $G = G(p_3, p_4)$ are scalar functions of the 4-momenta p_i.

(d) After some algebra, one finds that the differential decay rate $d\Gamma$ in the muon rest frame is equal to:

$$d\Gamma = \frac{\langle |\mathcal{M}|^2 \rangle c}{(4\pi)^4 \hbar m_\mu} dE_2 \frac{d^3\mathbf{p}_4}{E_4^2}, \tag{2}$$

where c is the speed of light, $\hbar$ is Planck's reduced constant, m_μ is the muon mass, E_2 represents the electron-antineutrino ($\bar{\nu}_e$) energy, $\mathbf{p}_4$ is the momentum of the electron, and E_4 represents the electron energy. If you assume that the electron and neutrino masses can effectively be set to zero in the problem, what is the resulting maximum energy E_2 and the maximum energy E_4 expressed in terms of the muon mass? Explain your reasoning.

(e) Working with the above expression, one finally ends up with the following energy distribution for the differential decay rate:

$$\frac{d\Gamma}{dE} = \left(\frac{g_W}{M_W c}\right)^4 \frac{m_\mu^2 E^2}{2\hbar (4\pi)^3} \left(1 - \frac{4E}{3 m_\mu c^2}\right). \tag{3}$$

Assume that the quantity E can take values from 0 to R (where R is a constant) and write down an explicit expression for the muon lifetime τ.

Formulae that may be useful. The meaning of the symbols is assumed to be known.

$$\bar{\psi} \equiv \psi^{\dagger}\gamma^{0},\ \gamma^{5} \equiv \mathrm{i}\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3},\ \bar{\Gamma} \equiv \gamma^{0}\Gamma^{\dagger}\gamma^{0},\ \sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^{\mu}p_{\mu}+mc),\ \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^{\mu}p_{\mu}-mc),$$
$$(\gamma^{0})^{\dagger} = \gamma^{0},\ (\gamma^{0})^{2} = 1,\ \gamma^{\nu} = \gamma^{0}(\gamma^{\nu})^{\dagger}\gamma^{0}. \tag{4}$$

Internal lines for some particles: $-\mathrm{i}g_{\mu\nu}/q^{2}, \frac{-\mathrm{i}(g_{\mu\nu}-q_{\mu}q_{\nu}/M^{2}c^{2})}{q^{2}-M^{2}c^{2}}$.

Some vertex factors for various interactions: $\mathrm{i}g_{e}\gamma^{\mu}, -\frac{\mathrm{i}g_{W,Z}}{2\sqrt{2}}\gamma^{\mu}(1-\gamma^{5})$.

A Lagrangian describing some type of field: $\mathcal{L} = \mathrm{i}\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^{2}\bar{\psi}\psi$.

Trace theorems: (below I use the notation $a' \equiv a^{\mu}\gamma_{\mu}$)

$$Tr(A+B) = Tr(A)+Tr(B),\ Tr(\alpha A) = \alpha Tr(A),\ Tr(ABC) = Tr(CAB) = Tr(BCA). \tag{5}$$

$$g_{\mu\nu}g^{\mu\nu} = 4,\ \{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu},\ a'b'+b'a' = 2ab. \tag{6}$$

$$\gamma_{\mu}\gamma^{\mu} = 4,\ \gamma_{\mu}\gamma^{\nu}\gamma^{\mu} = -2\gamma^{\nu},\ \gamma_{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\mu} = 4g^{\nu\lambda}. \tag{7}$$

$$\gamma_{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}\gamma^{\mu} = -2\gamma^{\sigma}\gamma^{\lambda}\gamma^{\nu},\ \gamma_{\mu}a'\gamma^{\mu} = -2a'. \tag{8}$$

$$\gamma_{\mu}a'b'\gamma^{\mu} = 4ab,\ \gamma_{\mu}a'b'c'\gamma^{\mu} = -2c'b'a'. \tag{9}$$

$$Tr(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu},\ Tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}). \tag{10}$$

$$Tr(a'b') = 4ab,\ Tr(a'b'c'd') = 4[(ab)(cd)-(ac)(bd)+(ad)(bc)], \gamma^{5} = \mathrm{i}\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}. \tag{11}$$

$$Tr(\gamma^{5}) = 0,\ Tr(\gamma^{5}\gamma^{\mu}\gamma^{\nu}) = 0,\ Tr(\gamma^{5}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\sigma}) = 4\mathrm{i}\varepsilon^{\mu\nu\lambda\sigma}. \tag{12}$$

$$Tr(\gamma^{5}a'b') = 0,\ Tr(\gamma^{5}a'b'c'd') = 4\mathrm{i}\varepsilon^{\mu\nu\lambda\sigma}a_{\mu}b_{\nu}c_{\lambda}d_{\sigma}, \tag{13}$$

where $\varepsilon^{\mu\nu\lambda\sigma}$ is -1 if $\mu\nu\lambda\sigma$ is an even permutation of 0123, +1 if it is an odd permutation, 0 if any two indices are the same. Finally, the trace over an odd number of γ matrices is zero. One has that $\varepsilon^{\mu\nu\lambda\sigma}\varepsilon_{\mu\nu\kappa\tau} = -2(\delta^{\lambda}_{\kappa}\delta^{\sigma}_{\tau} - \delta^{\lambda}_{\tau}\delta^{\sigma}_{\kappa})$ where δ^{μ}_{ν} is the Kronecker delta-function (equal to 1 if $\mu = \nu$, 0 otherwise)