Examination FY3403 Particle physics
Monday December 14, 2009
Solutions

1a) The total cross section is the same in the laboratory frame and in the CM frame.

Intuitively, a cross section in either of these two frames is an area perpendicular to the relative velocity of the two particles, hence also perpendicular to the relative velocity of the two frames. Perpendicular distances and perpendicular areas are invariant under the Lorentz transformation from one frame to the other.

To argue more formally, the probability of a two-particle reaction, or in practice the expected number of reactions in an experiment where many particles hit a target particle, is Lorentz invariant. The number of reactions is computed theoretically as a cross section times the number of incoming particles per area. The number of particles per area is Lorentz invariant, since the number of particles and the area are both Lorentz invariant. Since everything else is Lorentz invariant, we conclude that the cross section is Lorentz invariant.

1b) Write I for the reaction $e^+ + e^- \to \gamma + \gamma$ and II for the reverse reaction $\gamma + \gamma \to e^+ + e^-$.

Write the differential cross section for $e^+ + e^- \to \gamma + \gamma$, in the CM frame, including the statistical factor $S = 1/2$, as

$$\frac{\mathrm{d}\sigma(\mathrm{I})}{\mathrm{d}\Omega} = \frac{1}{2} \left(\frac{\hbar c}{8\pi E_{\mathrm{CM}}} \right)^2 \frac{|\vec{p}_\gamma|}{|\vec{p}_e|} \langle |\mathcal{M}(\mathrm{I})|^2 \rangle \ .$$

Similarly, write the differential cross section for $\gamma + \gamma \to e^+ + e^-$, in the CM frame, now with a statistical factor of $S = 1$, as

$$\frac{\mathrm{d}\sigma(\mathrm{II})}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi E_{\mathrm{CM}}} \right)^2 \frac{|\vec{p}_e|}{|\vec{p}_\gamma|} \langle |\mathcal{M}(\mathrm{II})|^2 \rangle \ .$$

The centre of mass energy E_{CM}, the electron (or positron) momentum $|\vec{p}_e|$ and the photon momentum $|\vec{p}_\gamma|$ are the same in the reactions I and II. The scattering angles θ and φ are also the same in the solid angle element $\mathrm{d}\Omega = \mathrm{d}(\cos\theta)\,\mathrm{d}\varphi$ and in the scattering amplitude $\mathcal{M}$.

In both reactions we have that

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{\mathrm{spins}} |\mathcal{M}|^2 \ .$$

The factor of 1/4 is there because we average over 4 spin states in the initial state (two spin states for the electron and two for the positron in reaction I, two spin states for each photon in reaction II).

Time reversal invariance implies that $\sum_{\mathrm{spins}} |\mathcal{M}(\mathrm{I})|^2 = \sum_{\mathrm{spins}} |\mathcal{M}(\mathrm{II})|^2$, so that

$$\frac{\mathrm{d}\sigma(\mathrm{II})}{\mathrm{d}\Omega} = 2\, \frac{\mathrm{d}\sigma(\mathrm{I})}{\mathrm{d}\Omega}\, \frac{|\vec{p}_e|^2}{|\vec{p}_\gamma|^2} \ .$$

The same relation holds for the total cross sections, after we integrate over angles,

$$\sigma(\mathrm{II}) = 2\,\sigma(\mathrm{I})\,\frac{|\vec{p}_e|^2}{|\vec{p}_\gamma|^2} \;.$$

In the CM frame all four particles have the same energy. The energy of each photon is $E_\gamma = |\vec{p}_\gamma|\,c$, and the energy of the electron and of the positron is the same, $E_e = E_\gamma = |\vec{p}_\gamma|\,c$. The velocity of the electron or the positron is

$$|\vec{v}_e| = \frac{|\vec{p}_e|\,c^2}{E_e} = \frac{|\vec{p}_e|\,c}{|\vec{p}_\gamma|} \;.$$

In the given formula for the cross section of reaction I the relative velocity $|\vec{v}|$ of the electron and positron is twice the velocity of each particle, $|\vec{v}| = 2\,|\vec{v}_e|$. Thus,

$$\sigma(I) = \frac{\pi {r_e}^2 c}{2\,|\vec{v}_e|} \;, \qquad \sigma(\mathrm{II}) = 2\,\sigma(\mathrm{I})\,\frac{|\vec{v}_e|^2}{c^2} = \frac{\pi {r_e}^2 |\vec{v}_e|}{c} \;.$$

1c) An estimate of the average photon energy:

$$E_{\gamma\mathrm{CMB}} = kT_{\mathrm{CMB}} = 8.617\times 10^{-5}\ \mathrm{eV/K}\times 2.73\ \mathrm{K} = 2.35\times 10^{-4}\ \mathrm{eV}\;.$$

Let $p_1 = (E_1/c, \vec{p}_1)$ and $p_2 = (E_2/c, \vec{p}_2)$ be the four-momenta of the two colliding photons. The centre of mass energy E_{CM} is given by the formula

$$E_{\mathrm{CM}}^2 = (p_1+p_2)^2c^2 = ({p_1}^2 + {p_2}^2 + 2\,p_1\cdot p_2)\,c^2 = 2\,p_1\cdot p_2\,c^2 = 2\,E_1E_2\,(1-\cos\alpha)\;,$$

where α is the angle between the three-momenta $\vec{p}_1$ and $\vec{p}_2$. The photons have mass zero, so that ${p_1}^2 = {p_2}^2 = 0$.

The most favourable situation for the production of an electron–positron pair is a head on collision, with $\cos\alpha = -1$, then $E_{\mathrm{CM}} = 2\sqrt{E_1E_2}$. The threshold centre of mass energy is $2m_ec^2$, where m_e is the electron mass. Then for $E_2 = 2.35\times 10^{-4}$ eV and a head on collision we have the threshold value

$$\begin{aligned} E_1 &= \frac{4(m_ec^2)^2}{4E_2} = \frac{(0.511\ \mathrm{MeV})^2}{2.35\times 10^{-4}\ \mathrm{eV}} \\ &= 1.11\times 10^{15}\ \mathrm{eV} = 1.11\times 10^{6}\ \mathrm{GeV} = 1.11\ \mathrm{PeV} = 1.11\ \mathrm{petaelectronvolt}\;. \end{aligned}$$

Strictly speaking the threshold is a little lower, because the average photon energy is $2.7\,kT$ rather than kT, and because many photons have energies above the average value.

1d) The number density of photons is $n = 400/\mathrm{cm}^3$. Inside a cube of volume d^3 there are nd^3 photons, presenting a target of total area $A = nd^3\sigma$, with $\sigma = \pi {r_e}^2$. The photons hitting the cube are spread over an area of d^2, and the probability that a photon will react if it hits the cube, is

$$P(d) = \frac{A}{d^2} = nd\sigma\;.$$

We have assumed in this calculation that $P(d)$ is small, so that we can neglect the probability that one photon in the box will shadow for another.

The probability that a high energy photon is able to travel a small distance d without interacting, is $1 - P(d) = 1 - nd\sigma$. The probability that it is able to travel a large distance $D = Nd$ without interacting, is

$$1 - P(D) = (1 - nd\sigma)^N = \mathrm{e}^{-Nnd\sigma} = \mathrm{e}^{-nD\sigma} \ .$$

The probability that it travels a distance $x = Nd$ without interacting, and then interacts in the next small interval $\mathrm{d}x = d$, is

$$\mathrm{d}P = \mathrm{e}^{-n\sigma x} n\sigma \, \mathrm{d}x \ .$$

The total probability that it interacts somewhere is

$$\int_0^\infty \mathrm{e}^{-n\sigma x} n\sigma \, \mathrm{d}x = \int_0^\infty \mathrm{e}^{-y} \, \mathrm{d}y = 1 \ ,$$

with $y = n\sigma x$. The average distance it travels before interacting is

$$\begin{aligned} \overline{D} &= \int_0^\infty x \, \mathrm{e}^{-n\sigma x} n\sigma \, \mathrm{d}x = \frac{1}{n\sigma} \int_0^\infty y \, \mathrm{e}^{-y} \, \mathrm{d}y = \frac{1}{n\sigma} \\ &= \frac{1}{400 \ \mathrm{cm}^{-3} \times \pi \times (2.818 \times 10^{-15} \mathrm{m})^2} = 1.00 \times 10^{20} \ \mathrm{m} \ . \end{aligned}$$

This is about 10 000 lightyears, one third of the distance to the centre of our Milky Way galaxy. Photons of such high energies could not reach us from other galaxies.

2a) Possible values are $\ell = 0, 1, 2, 3, \ldots$ and $s = 0, 1$.

Two particles can form a bound state when they attract each other. Typically, the attraction is stronger the closer they come together. In a quantum state with relative angular momentum ℓ the wave function goes as r^ℓ when $r \to 0$, where r is the distance between the particles. To minimize the energy we should minimize ℓ in order to bring the particles close together, therefore the ground state usually has $\ell = 0$.

A quark–antiquark bound state has total angular momentum $\vec{J} = \vec{L} + \vec{S}$, where $\vec{L}$ is the relative orbital angular momentum and $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the sum of the two spins. In the ground state we may assume that $\ell = 0$, this means that $\vec{L} = 0$ and the total angular momentum is equal to the total spin: $\vec{J} = \vec{S}$.

The quark model prediction is that π^0 and η are superpositions of $u\overline{u}$ and $d\overline{d}$, in both cases with quantum numbers $\ell = 0$ and $s = 0$, giving $P = (-1)^{\ell+1} = -1$ and $C = (-1)^{\ell+s} = +1$.

Similarly, ρ^0 and ω are superpositions of $u\overline{u}$ and $d\overline{d}$ with $\ell = 0$ and $s = 1$, giving $P = (-1)^{\ell+1} = -1$ and $C = (-1)^{\ell+s} = -1$.

2b) A state of n photons has charge conjugation symmetry $C = (-1)^n$.

Parapositronium in the ground state, with $\ell = 0$ and $s = 0$, has $C = (-1)^{\ell+s} = +1$ and can decay into two photons.

Orthopositronium in the ground state, with $\ell = 0$ and $s = 1$, has $C = (-1)^{\ell+s} = -1$ and has to decay into three photons (C is conserved in electromagnetic interactions).

Each photon in the final state brings with it a factor of α, the fine structure constant, in the decay rate. For this reason (and for reasons of phase space) the three photon decay of orthopositronium is slower (has a longer life time) than the two photon decay of parapositronium.

2c) Decay rate:

$$\Gamma = \mathcal{L}\sigma = |\psi_{\rm rel}(0)|^2\, |\vec{v}|\, \frac{4\pi {r_e}^2 c}{|\vec{v}|} = \frac{1}{\pi}\left(\frac{\alpha m_e c}{2\hbar}\right)^3 4\pi c \left(\frac{\alpha\hbar}{m_e c}\right)^2 = \frac{\alpha^5 m_e c^2}{2\hbar}\ .$$

Life time:

$$\tau = \frac{1}{\Gamma} = \frac{2\hbar}{\alpha^5 m_e c^2} = 1.245 \times 10^{-10}\ \mathrm{s}\ .$$

3a) The particles of a generation can be produced in particle–antiparticle pairs. But this is not really a transformation between generations.

A quark of one generation may be transformed into a quark of another generation by the weak interaction, when the quark emits or absorbs a W boson.

Technically speaking, the weak interaction mixes the three quarks d, s, b of the three generations, as described by the unitary Kobayashi–Maskawi (or Cabibbo–Kobayashi–Maskawi, CKM) matrix.

A neutrino of one generation may change into a neutrino of another generation by neutrino oscillation.

Technically speaking again, the neutrino oscillations are due to a mass matrix which is non-diagonal in the generations. The unitary matrix transforming the three neutrino mass eigenstates into the neutrinos of the three generations is called the Maki–Nakagawa–Sakata (MNS) matrix.

3b) The coupling constant in QCD is energy dependent, as a result of vacuum polarization, and decreases with increasing energy, so that the quarks behave asymptotically as non-interacting particles (but they do interact at any finite energy). This is called asymptotic freedom, it implies that QCD at high energy (a few GeV) can be treated by perturbation theory.

3c) A grand unified theory unifies both strong, electromagnetic and weak interactions, and has one single non-Abelian gauge group and one single coupling constant. The present standard model is a gauge theory with three gauge groups: SU(3) for QCD, SU(2) and U(1) for the electroweak interaction. It has three coupling constants, one for each gauge group. One possibility is to include these three groups as subgroups of SU(5).

The strongest argument in favour of grand unification is that the energy dependence of the three coupling constants is such that they seem to approach a common value at very high energies, around 10^{16} GeV = 10^{25} eV.

Another argument in favour of grand unification is that it could explain why electric charge is quantized, with the proton and electron charges cancelling exactly, so that neutral atoms have exactly zero net charge. We know of no other way to explain this experimental fact (if magnetic monopoles exist, that would be another explanation).

The strongest argument against proposed grand unified theories is that they predict proton decay, which has not been observed.

Another argument against is that the three curves showing the energy dependence of the coupling constants do not meet exactly. Supersymmetry is claimed to be able to solve this problem, by bringing in new particles at energies above what is available in present accelerators.