NTNU

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Solution to the exam in FY3403 PARTICLE PHYSICS

Friday December 2, 2011

This solution consists of 7 pages.

Problem 1. Quark model for baryons

Give a qualitative description of how one in the quark model assumes that baryons are made of quarks. In particular try to explain

a) how many quarks and antiquarks the baryons are made of,

In the crudest approximation a baryon is composed of three quarks and a anti-baryon is composed of three antiquarks. The quantum numbers for the (anti-)baryon is determined from this *valence quark* model.

Note: In a more accurate description we must take into account that the baryon contains a fluctuating number of quark-antiquark pairs (called sea quarks), and also a fluctuating number of *gluons*.

b) which spin S the total baryon system may have,

Since the quarks each have spin $S = \frac{1}{2}$ it follows that the total spin of the baryon is $S = \frac{1}{2}$ or $S = \frac{3}{2}$.

Note: In addition the quarks in a baryon may have an angular momentum L. We then can find baryons with spin $J = |L - S|, \ldots, L + S$. When L varies this states form *Regge trajectories*.

c) which isospin I the total baryon system may have,

If the baryon only consists of u or d quarks the isospin will be $I = \frac{1}{2}$ or $I = \frac{3}{2}$.
If the baryon consists of two u/d quarks (and one $c/s/t/b$ quark), the isospin will be $I = 0$ or $I = 1$. If the baryon consists of one u/d quark (and two $c/s/t/b$ quarks), the isospin will be $I = \frac{1}{2}$. If the baryon have no u/d quarks the isospin will be $I = 0$.

d) why there are no baryons with charge -2 (in units of the positron charge),

The quarks have electric charge $Q = -\frac{1}{3}$ or $Q = +\frac{2}{3}$. Thus from three quarks we can only make $Q = -1$, $0\,1$, 2.

Note: It has been speculated that a *pentaquark*, with four valence quarks and a valence antiquark, might exist. A pentaquark baryon might have electric charge $Q = -2$.

Problem 2.

The normalized spin/flavor wave function for Δ^{++} with spin $S_z = \frac{3}{2}$ is given by

$$|\Delta^{++}\ \tfrac{3}{2}\rangle = |u\uparrow\rangle|u\uparrow\rangle|u\uparrow\rangle. \tag{1}$$

a) Find the normalized spin/flavor wave function for Δ^{+} with spin $S_z = \frac{3}{2}$.

We use the ladder operators for isospin $I^- = I_1^- + I_2^- + I_3^-$ on equation (1) and normalize by hand. This gives

$$|\Delta^{+}\ \tfrac{3}{2}\rangle = \sqrt{\tfrac{1}{3}}\left(|d\uparrow\rangle\,|u\uparrow\rangle\,|u\uparrow\rangle + |u\uparrow\rangle\,|d\uparrow\rangle\,|u\uparrow\rangle + |u\uparrow\rangle\,|u\uparrow\rangle\,|d\uparrow\rangle\right). \tag{2}$$

b) Find the normalized spin/flavor wave function for Δ^{+} with spin $S_z = \frac{1}{2}$.

We use the ladder operators for spin $S^- = S_1^- + S_2^- + S_3^-$ on equation (2) and normalize by hand. This gives

$$\begin{aligned}|\Delta^{+}\ \tfrac{1}{2}\rangle &= \frac{1}{3}\left(|d\downarrow\rangle\,|u\uparrow\rangle\,|u\uparrow\rangle + |d\uparrow\rangle\,|u\downarrow\rangle\,|u\uparrow\rangle + |d\uparrow\rangle\,|u\uparrow\rangle\,|u\downarrow\rangle\right)\\ &+ \frac{1}{3}\left(|u\downarrow\rangle\,|d\uparrow\rangle\,|u\uparrow\rangle + |u\uparrow\rangle\,|d\downarrow\rangle\,|u\uparrow\rangle + |u\uparrow\rangle\,|d\uparrow\rangle\,|u\downarrow\rangle\right)\\ &+ \frac{1}{3}\left(|u\downarrow\rangle\,|u\uparrow\rangle\,|d\uparrow\rangle + |u\uparrow\rangle\,|u\downarrow\rangle\,|d\uparrow\rangle + |u\uparrow\rangle\,|u\uparrow\rangle\,|d\downarrow\rangle\right).\end{aligned} \tag{3}$$

c) The magnetic moment of a baryon with spin/flavor wave function $|\Psi\rangle$ is defined as

$$\mu_z = \langle\Psi\,|\sum_i \frac{e\,\boldsymbol{Q}_i}{2\boldsymbol{m}_i}\,\boldsymbol{S}_{iz}\,|\,\Psi\rangle, \tag{4}$$

where the sum is over the three positions in the wave function. (Note that $\boldsymbol{Q}_i$, $\boldsymbol{m}_i$ and $\boldsymbol{S}_{iz}$ is *operators* which take different values depending on the states they act on.)
Find the magnetic moment of Δ^{++} with spin $S_z = \frac{3}{2}$. Assume that $m_u = m_d$.

Here we use *natural units*, i.e. units where $\hbar = c = 1$ (Eq. 4 only valid in natural units).

$$\mu_z = \left(\frac{2}{3}\times\frac{1}{2} + \frac{2}{3}\times\frac{1}{2} + \frac{2}{3}\times\frac{1}{2}\right)\left(\frac{e}{2m_u}\right) = \frac{e}{2m_u} \tag{5}$$

d) Find the magnetic moment of Δ^{+} with spin $S_z = \frac{1}{2}$. Assume that $m_u = m_d$.

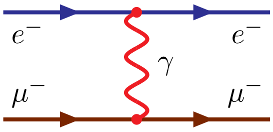
$$\begin{aligned}\mu_z &= \tfrac{1}{9}\left[\left(-\tfrac{1}{3}\times -\tfrac{1}{2} + \tfrac{2}{3}\times\tfrac{1}{2} + \tfrac{2}{3}\times\tfrac{1}{2}\right)\times 3 + \left(-\tfrac{1}{3}\times\tfrac{1}{2} + \tfrac{2}{3}\times -\tfrac{1}{2} + \tfrac{2}{3}\times\tfrac{1}{2}\right)\times 6\right]\left(\tfrac{e}{2m_u}\right)\\ &= \frac{e}{12m_u}\end{aligned} \tag{6}$$

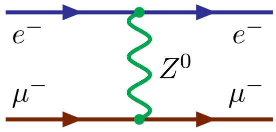
Hint to point a–b): Use the ladder operators.

Problem 3. Interaction processes and Feynman diagrams

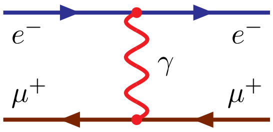
Draw lowest order Feynman diagrams for the following interaction processes (if the process is possible)

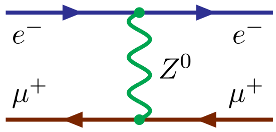
a) $e^- + \mu^- \longrightarrow e^- + \mu^-$





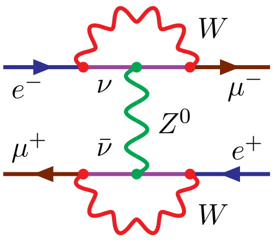
b) $e^- + \mu^+ \longrightarrow e^- + \mu^+$





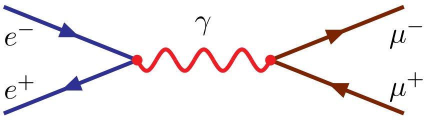
c) $e^- + \mu^+ \longrightarrow e^+ + \mu^-$

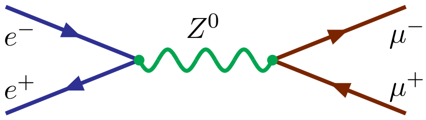
It is essentially correct to say that this process is impossible, due to conservation of electron and muon numbers. However, with the possibility of neutrino oscillations there should be an exceedingly tiny amplitude for the process to occur through radiative corrections



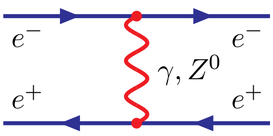
An educated guess is that this amplitude is of order $\left(\Delta m_{21}^2/M_W^2\right)^2$ relative to the amplitude for the amplitude for $\nu + \bar{\nu} \to \nu + \bar{\nu}$ scattering, and thus that the cross section is of order $\left(\Delta m_{21}^2/M_W^2\right)^4 \approx 10^{-52}$ relative to the $\nu\bar{\nu}$ scattering cross section (which is already very small). Not much chance of observing this process during the lifetime of our universe!

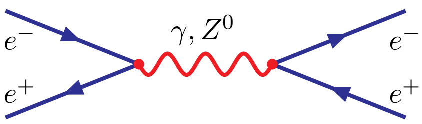
d) $e^- + e^+ \longrightarrow \mu^+ + \mu^-$





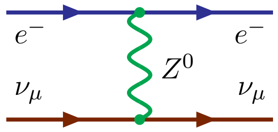
e) $e^- + e^+ \longrightarrow e^+ + e^-$



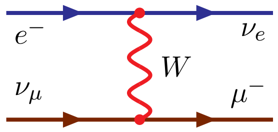


The notation indicates that there are actually four diagrams, two with γ exchange and two with Z^0 exchange.

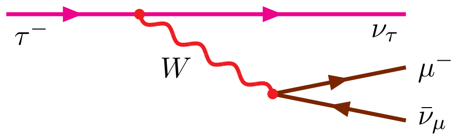
f) $e^- + \nu_\mu \longrightarrow e^- + \nu_\mu$



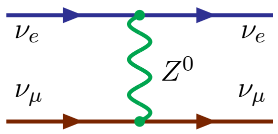
g) $e^- + \nu_\mu \longrightarrow \nu_e + \mu^-$



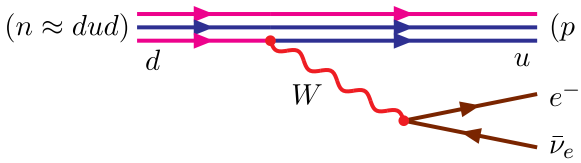
h) $\tau^- \longrightarrow \mu^- + x$ (replace x by some possible set of particles)



i) $\nu_e + \nu_\mu \longrightarrow \nu_\mu + \nu_e$

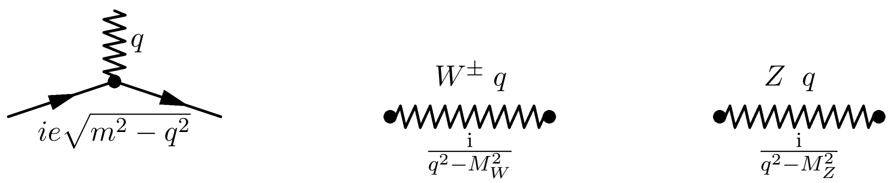


j) $n \longrightarrow p + x$ (replace x by some possible set of particles)

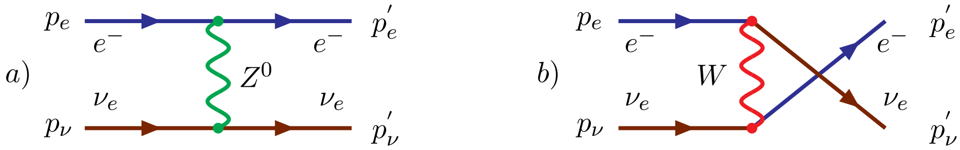


Problem 4. Elastic $\nu_e + e^- \to \nu_e + e^-$ scattering

Assume Feynman rules as *indicated* below, where m is the mass of the heaviest fermion involved in the interaction vertex and q is the four-momentum of the virtual messenger particle.



a) Draw all lowest order Feynman diagrams for the process.



Here we have added the 4-momentum to the fermion lines.

b) Write down the corresponding algebraic expressions for the scattering amplitude $\mathcal{M}_{fi}$.

We do the calculations i the center of mass system (CM) since equation (20) is derived (and only valid) in CM. We set the neutrino mass to zero (ν_e is not a mass eigenstate anyhow). Since this is elastic scattering we have that $E_e = E_e'$, and $E_\nu = E_\nu' = |\mathbf{p}_e| = |\mathbf{p}_\nu| = |\mathbf{p}_e'| = |\mathbf{p}_\nu'|$.

$$\mathcal{M}_{fi}^a = e^2 \frac{\sqrt{-q_Z^2}\sqrt{m_e^2 - q_Z^2}}{q_Z^2 - M_Z^2} \tag{7}$$

$$\mathcal{M}_{fi}^b = e^2 \frac{m_e^2 - q_W^2}{q_W^2 - M_W^2} \tag{8}$$

where

$$q_Z^2 = (p_e - p_e')^2 = -4E_\nu^2 \sin^2(\theta/2) \tag{9}$$

$$\begin{aligned} q_W^2 &= (p_e - p_\nu')^2 = m_e^2 - 2E_\nu E_e - 2E_\nu^2 \cos\theta \\ &= m_e^2 - 2E_\nu(E_e + E_\nu) + 4E_\nu^2 \sin^2(\theta/2), \end{aligned} \tag{10}$$

where θ is the scattering angle, $\mathbf{p}_e \cdot \mathbf{p}_e' = \mathbf{p}_\nu \cdot \mathbf{p}_\nu' = E_\nu^2 \cos\theta$, and $\mathbf{p}_e \cdot \mathbf{p}_\nu' = E_\nu^2 \cos(\pi - \theta)$. Note that both amplitudes are real.

c) Find the total scattering cross-section. You may assume that $|q^2| \ll M_W^2$ and $|q^2| \ll M_Z^2$ to simplify expressions.

Here we have that $S = 1$, $|\mathbf{p}_f| = |\mathbf{p}_i|$ and there is no ϕ-dependence giving

$$\sigma = \frac{1}{64\pi^2(E_e + E_\nu)^2} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \, |\mathcal{M}_{fi}|^2 = \frac{1}{8\pi E^2} \int_0^1 u\mathrm{d}u \left(\mathcal{M}_{fi}^a + \mathcal{M}_{fi}^b\right)^2 \tag{11}$$

where we have introduced the variable $u = \sin(\theta/2)$, and $E = E_e + E_\nu = \sqrt{s}$, the total energy in CM. Assuming that $|q_W^2| \ll M_W^2$ and $|q_Z^2| \ll M_Z^2$ we have

$$\begin{aligned} \mathcal{M}_{fi} = \mathcal{M}_{fi}^a + \mathcal{M}_{fi}^b &\approx -\frac{2e^2 E_\nu}{M_Z^2} u\sqrt{m_e^2 + 4E_\nu^2 u^2} - \frac{2e^2 E_\nu}{M_W^2}\left(E - 2E_\nu u^2\right) \\ &= -\frac{8\pi\alpha E E_\nu}{M_W^2}\left[1 + \frac{M_W^2}{M_Z^2} u \sqrt{\frac{m_e^2}{E^2} + \frac{4E_\nu^2}{E^2}u^2} - \frac{2E_\nu}{E}u^2\right] \end{aligned} \tag{12}$$

First let us consider the case where $E_\nu \ll m_e$. In this case the electron is non-relativistic ($E \approx E_e \approx m_e$), and we have

$$\mathcal{M}_{fi} \approx -\frac{8\pi\alpha m_e E_\nu}{M_W^2}\left(1 + \frac{M_W^2}{M_Z^2}u\right) \tag{13}$$

and

$$\sigma \approx 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\left(\frac{1}{2}+\frac{2}{3}\frac{M_W^2}{M_Z^2}+\frac{1}{4}\frac{M_W^4}{M_Z^4}\right) \approx 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\, 1.17 \tag{14}$$

Next consider the case where $m_e \ll E_\nu \ll M_W$ giving $E \approx 2E_\nu$. In this case we have

$$\mathcal{M}_{fi} \approx -\frac{8\pi\alpha E E_\nu}{M_W^2}\left[1-\left(1-\frac{M_W^2}{M_Z^2}\right)u^2\right] \tag{15}$$

and

$$\begin{aligned}\sigma &\approx 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\left[\frac{1}{2}-\frac{2}{4}\left(1-\frac{M_W^2}{M_Z^2}\right)+\frac{1}{6}\left(1-\frac{M_W^2}{M_Z^2}\right)^2\right] && (16)\\ &= 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\left(\frac{1}{6}+\frac{1}{6}\frac{M_W^2}{M_Z^2}+\frac{1}{6}\frac{M_W^4}{M_Z^4}\right) \approx 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\, 0.40 && (17)\end{aligned}$$

In the more general case (we still have $E_\nu \ll M_W$) we have

$$\begin{aligned}\sigma &\approx 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\int_0^1 udu\left(1+\frac{M_W^2}{M_Z^2}u\sqrt{\frac{m_e^2}{E^2}+\frac{4E_\nu^2}{E^2}u^2}-\frac{2E_\nu}{E}u^2\right)^2\\ &= 8\pi\alpha^2 \frac{E_\nu^2}{M_W^4}\left[\frac{1}{2}-\frac{E_\nu}{E}+\frac{2E_\nu^2}{3E^2}+\frac{M_Z^2}{M_W^2}f(E_\nu)+\frac{M_Z^4}{M_W^4}\left(\frac{m_e^2}{E^2}+\frac{4E_\nu^2}{6E^2}\right)\right] && (18)\end{aligned}$$

where

$$\begin{aligned}f(E_\nu) &= \frac{2m_e}{E}\int_0^1 duu^2\left(1-\frac{2E_\nu}{E}u^2\right)\sqrt{1+\frac{4E_\nu^2}{m_e^2}u^2}\\ &= -\frac{1}{32}\frac{m_e^2}{E_\nu^2}\sqrt{1+\frac{4E_\nu^2}{m_e^2}}+\frac{1}{16}\frac{m_e^2}{E_\nu^2}\left(1+\frac{4E_\nu^2}{m_e^2}\right)^{3/2}+\frac{1}{64}\frac{m_e^4}{EE_\nu^3}\left(1+\frac{4E_\nu^2}{m_e^2}\right)^{3/2}\\ &\quad -\frac{1}{12}\frac{m_e^2}{EE_\nu}\left(1+\frac{4E_\nu^2}{m_e^2}\right)^{3/2}-\frac{1}{128}\frac{m_e^4}{EE_\nu^3}\sqrt{1+\frac{4E_\nu^2}{m_e^2}}\\ &\quad -\frac{1}{256}\frac{m_e^5}{EE_\nu^4}\ln\left(\frac{2E_\nu}{m_e}+\sqrt{1+\frac{4E_\nu^2}{m_e^2}}\right)-\frac{1}{64}\frac{m_e^3}{E_\nu^3}\ln\left(\frac{2E_\nu}{m_e}+\sqrt{1+\frac{4E_\nu^2}{m_e^2}}\right) && (19)\end{aligned}$$

This seemingly complicated function is depicted in Figure 1.

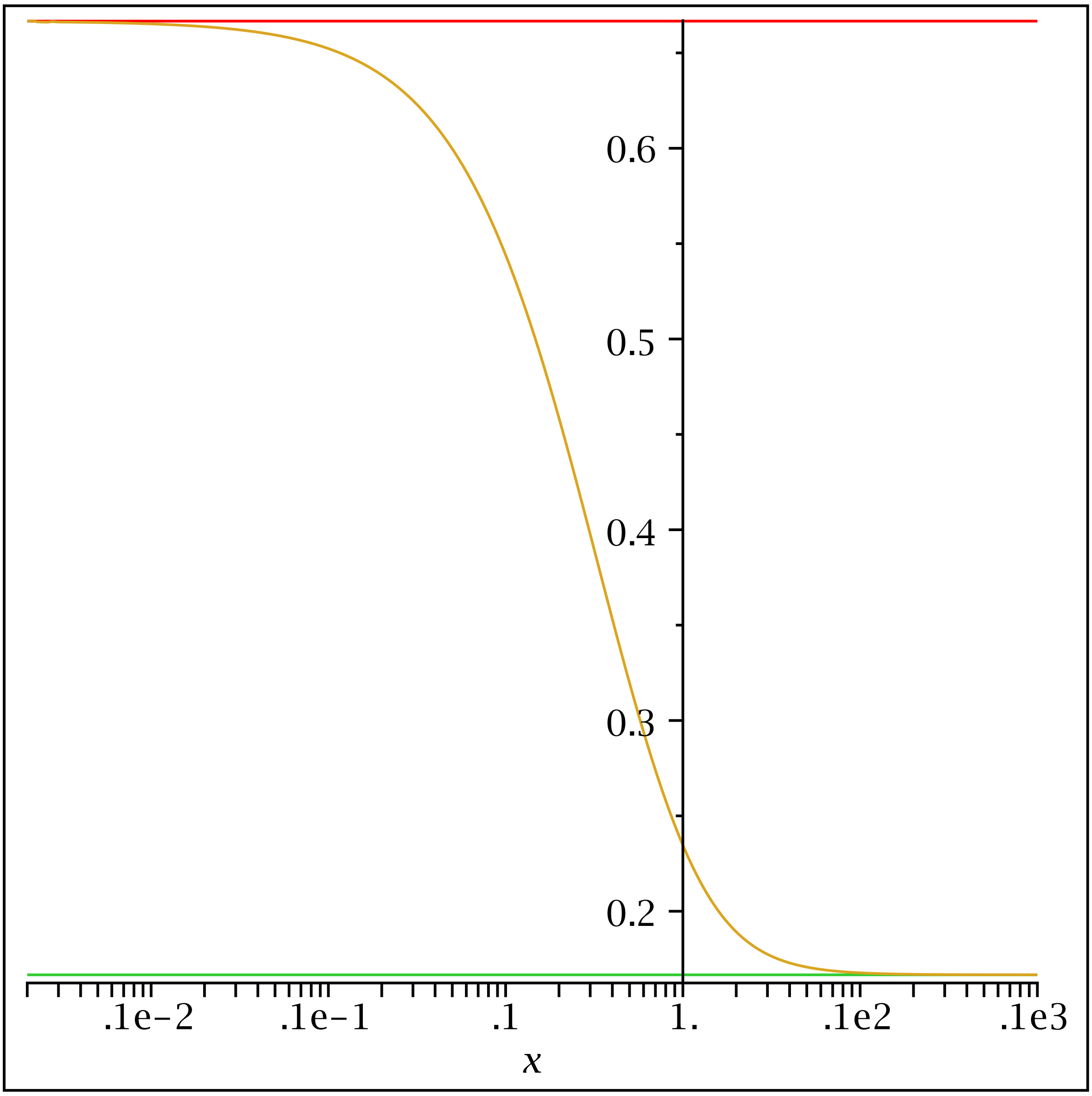


Figure 1: Plot of $f(\frac{E_\nu}{m_e})$. The asymptotic values, $\frac{2}{3}$ and $\frac{1}{6}$, are included in the plot, red and green line respectively. Note that the x-axes is logarithmic spanning $[2 \cdot 10^{-4}, 10^2]$, so the function is slowly varying from $\frac{2}{3}$ for $\frac{E_\nu}{m_e} \approx 10^{-3}$ and smaller, to $\frac{1}{6}$ for $\frac{E_\nu}{m_e} \approx 10$ and larger.

Given: We use *natural units*, i.e. units where $\hbar = c = 1$. The connection between scattering amplitude $\mathcal{M}_{fi}$ and scattering cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S}{64\pi^2} \frac{|\mathcal{M}_{fi}|^2}{(E_1 + E_2)^2} \frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}. \tag{20}$$