NTNU Trondheim, Institutt for fysikk

Examination for FY3403 Particle Physics
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Allowed tools: mathematical tables, pocket calculator
Some formulas and data can be found on p. 2ff.

1. Chirality vs. helicity.
a.) Helicity for a free, massive particle is (3 pts)
☐ frame dependent
☐ conserved
b.) Chirality for a free, massive particle is (3 pts)
☐ frame dependent
☐ conserved

2. $\mathrm{SU}(2)_L$.
Consider one generation of quarks before spontanous symmetry breaking, i.e. when they are massless.
a.) How do the quarks transform under $\mathrm{SU}(2)_L$ transformations? (5 pts)
b.) Show that the Lagrangian describing these fields (without interactions) is invariant under global, but not under local $\mathrm{SU}(2)_L$ transformations. (6 pts)

a.) We have to distinguish left- and right-chiral fields,

$$\psi_L = P_L\psi = \frac{1}{2}(1-\gamma^5)\psi \quad \text{and} \quad \psi_R = P_R\psi = \frac{1}{2}(1+\gamma^5)\psi\,.$$

Left-chiral quark fields live in the fundamental representation of $\mathrm{SU}(2)_L$, i.e. $Q = (u,d)_L^T$, while right-chiral fields are singlets, i.e. u_R and d_R. Singlets are invariant under $\mathrm{SU}(2)_L$ transformation, while the doublets tranform as

$$Q \to Q' = \exp\left\{\frac{\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right\} Q$$

where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \alpha_3\}$ parameterise the transformation, and the generators $\boldsymbol{T} = \boldsymbol{\tau}/2$ are given by the Pauli matrices.

b.) Since the right-chiral fields are singlets, we have to consider only the left-chiral fields. Their Lagrangian

$$\mathscr{L} = \mathrm{i}\bar{Q}\partial\!\!\!/ Q$$

transform under a global transformation as

$$\mathscr{L} \to \mathscr{L}' = \mathrm{i}\bar{Q}U^\dagger\partial\!\!\!/ UQ = \mathrm{i}\bar{Q}\partial\!\!\!/ Q = \mathscr{L}\,,$$

since $U^\dagger U = 1$ and the parameters $\boldsymbol{\alpha}$ are constant. In case of a local transformation, the derivative produce an additional term,

$$\mathscr{L} \to \mathscr{L} + \mathrm{i}\bar{Q}\gamma^\mu U^\dagger(\partial_\mu U)Q$$

which breaks the local invariance.

3. Polarization sum for a massive spin-1 particle.
a.) Write down a possible set of polarisation vectors $\varepsilon_\mu^{(r)}$, $r = 1, \ldots, n$ for a massive spin-1 particle. (4 pts)
b.) Derive the completeness relation for a massive spin-1 particle, (6 pts)

$$\sum_{r=1}^{n} \varepsilon_\mu^{(r)} \varepsilon_\nu^{(r)*} = -\eta_{\mu\nu} + k_\mu k_\nu / m^2 \,.$$

A massive vector field A^μ has four components in $d = 4$ space-time dimensions, while it has only $2s + 1 = 3$ independent spin components. Correspondingly, a four-vector A^μ transforms under a rotation as $(A^0, \boldsymbol{A})$, i.e. it contains a scalar and a three-vector. Thus the physical components of a massive spin-1 field in its rest-frame are given by $(0, \boldsymbol{A})$. We can choose the three polarisation vectors in the rest frame e.g. as the Cartesian unit vectors, $\boldsymbol{\varepsilon}_i \propto \mathbf{e}_i$. They satisfy $\varepsilon_\mu^{(r)} \varepsilon^{\mu(r)} = -1$ and, since in the rest-frame $k^\mu = (m, \mathbf{0})$ also $k_\mu \varepsilon_\mu^{(r)} = 0$. Next we evaluate

$$\sum_r \varepsilon^{\mu(r)} \varepsilon^{\nu(r)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Next we have to express this using the relevant tensors, here $\eta^{\mu\nu}$ and $k^\mu k^\nu / m^2$, where we divided by m^2 to get the right dimension. This gives for $k^\mu = (m, \mathbf{0})$

$$\sum_r \varepsilon^{\mu(r)} \varepsilon^{\nu(r)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -\eta^{\mu\nu} + k^\mu k^\nu / m^2 \tag{2}$$

If we are not able to guess this, we can derive it formally: We set

$$\sum_r \varepsilon^{\mu(r)} \varepsilon^{\nu(r)} = A\eta^{\mu\nu} + Bk^\mu k^\nu / m^2 \,.$$

Asking then $k_\mu \varepsilon^{\mu(r)} = 0$, gives

$$0 = Ak^\nu + Bk^\nu$$

or $A = -B$. The normalisation condition requires $A = -1$.

4. W decays.
Calculate the squared matrix element $|\bar{\mathcal{M}}|^2$ for the decay rate of a W boson into a massless

fermion pair, summed over final state spins and averaged over intial ones. Express $|\bar{\mathcal{M}}|^2$ as function of invariant scalar products, $f(p_1 \cdot p_2, p_1 \cdot k, \dots)$, where k, p_1 and p_2 are the four-momenta of the three particles. (16 pts)

The Feynman amplitude for this process is

$$\mathrm{i}\mathcal{M} = -\frac{\mathrm{i}g}{2\sqrt{2}}\varepsilon_\mu^{(r)}(k)[\bar{u}(p_2)\gamma^\mu(1-\gamma^5 v(p_1)] \tag{3}$$

and thus

$$|\mathcal{M}|^2 = \frac{g^2}{8}\varepsilon_\mu^{(r)}\varepsilon_\nu^{(r)}[\bar{u}(p_2)\gamma^\mu(1-\gamma^5)v(p_1)][\bar{u}(p_2)\gamma^\nu(1-\gamma^5)v(p_1)]^* \,. \tag{4}$$

We determine $[\cdots]^* = [\cdots]^\dagger$ either directly

$$[u(p_2)^\dagger\gamma^0\gamma^\nu(1-\gamma^5)v(p_1)]^\dagger = [v(p_1)^\dagger(1-\gamma^5)\gamma^{\nu\dagger}\gamma^0 u(p_2)] = [v(p_1)^\dagger(1-\gamma^5)\gamma^0\gamma^\nu\gamma^0\gamma^0 u(p_2)] \tag{5}$$

$$= [v(p_1)^\dagger\gamma^0(1+\gamma^5)\gamma^\nu u(p_2)] = [\bar{v}(p_1)^\dagger\gamma^\nu(1-\gamma^5)u(p_2)] \,. \tag{6}$$

or we use $\overline{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0$,

$$\overline{\gamma}^5 = \gamma^0\gamma^{5\dagger}\gamma^0 = \gamma^0\gamma^5\gamma^0 = -\gamma^5 \,. \tag{7}$$

Next we sum over the spins of the fermions, setting $A^\mu = \gamma^\mu(1-\gamma^5)$,

$$\sum_{s_1,s_2}[\bar{u}_a(s_2,p_2)A^\mu_{ab}v_b(s_1,p_1)][\bar{v}_d(s_1,p_1)^\dagger A^\nu_{de}u_e(s_2,p_2)] = (\not{p}_2)_{ea}A^\mu_{ab}(\not{p}_1)_{bd}A^\nu_{de} = \mathrm{tr}[\not{p}_2 A^\mu \not{p}_1 A^\nu] \tag{8}$$

We anti-commute the right factor $(1-\gamma^5)$ to the left (or the left to the right), and use $(1-\gamma^5)^2 = 2(1-\gamma^5)$,

$$\mathrm{tr}[\not{p}_2\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)] = 2\,\mathrm{tr}[\not{p}_2\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu] \,. \tag{9}$$

The factor with γ^5 will lead to term containing the completely anti-symmetric Levi-Civita tensor, which is contracted with the symmetric factor $\varepsilon_\mu^{(r)}\varepsilon_\nu^{(r)}$. Thus this term vanishes. The remaining term gives

$$\mathrm{tr}[\not{p}_2\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)] = 2\,\mathrm{tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] = 8\,\mathrm{tr}[p_2^\mu p_1^\nu - (p_2\cdot p_1)\eta^{\mu\nu} + p_2^\nu p_1^\mu] \tag{10}$$

We average next over the polarisations of the W,

$$\frac{1}{3}\sum_r|\mathcal{M}|^2 = \frac{g^2}{3}(-\eta_{\mu\nu} + k_\mu k_\nu/M^2)[p_2^\mu p_1^\nu - (p_2\cdot p_1)\eta^{\mu\nu} + p_2^\nu p_1^\mu] \tag{11}$$

$$= \frac{g^2}{3}[(p_2\cdot p_1) + 2(k\cdot p_1)(k\cdot p_2)/M^2] \tag{12}$$

5. Neutrino scattering.

Let us assume that at the Galactic Center a source of high-energy neutrons exist. These neutrons escape from the source and produce neutrinos via beta-decay which we aim to

measure.
a.) Neglect neutrio masses and oscillations. Draw the lowest order Feynman diagrams for the interaction of these neutrinos with matter (assumed to consist of e^-, u- and d-quarks). (6 pts)
b.) In which of the diagrams can the virtual particle be on mass-shell? Find the required neutrino energy. (6 pts)
c.) Estimate the maximal cross section (in SI or cgs units) using the Breit-Wigner formula. (6 pts)
d.) Describe qualitatively, how neutrino mixing and oscillations would change this picture (maximal 100 words). (6 pts)

a.) s-channel Glashow resonance $\bar{\nu}_e e^- \to W^- \to$ all, t-channel CC $\bar{\nu}_e u \to e^+ d$ and t-channel NC $\bar{\nu}_e X \to \bar{\nu}_e X$ with $X = \{e^-, u, d\}$ interactions.

b.) The Mandelstam variables t and u are always negative, while s is positive. Thus only the denominator in the propagator of the first diagram, $\propto 1/(s - M^2)$ can become zero. In the lab frame

$$s = M^2 = (p_1 + p_2)^2 = 0 + 2E_\nu m_e + m_e^2$$

or

$$E_\nu = \frac{M^2 - m_e^2}{2m_e} \simeq \frac{M^2}{2m_e} \simeq 6 \times 10^{15}\,\text{eV}$$

c.) The prefactor of Breit-Wigner formula (13) becomes

$$\frac{4\pi s}{p_{\text{cms}}^2} \frac{(2s_r + 1)}{(2s_1 + 1)(2s_2 + 1)} = 16\pi\,\frac{3}{2} = 24\pi \tag{13}$$

where we use that only left-chiral neutrinos are produced. Thus

$$\sigma(\bar{\nu}_e e^- \to W \to \bar{f}f) = 24\pi\,\frac{\Gamma(W \to \bar{\nu}_e e^-)\Gamma(W \to \text{all})}{(s - M_W^2)^2 + (M_W\Gamma_W)^2} \tag{14}$$

$$= \frac{24\pi}{M_W^2}\,\text{BR}(W \to \bar{\nu}_e e^-)\text{BR}(\text{all}) \tag{15}$$

where BR denotes the branching ratio into the considered channel. We include all final states, and thus $\text{BR}(W \to \text{all}) = 1$. With $\text{BR}(W \to \bar{\nu}_e e^-) = 10.5\%$ it follows

$$\sigma(\bar{\nu}_e e^- \to W \to \bar{f}f) \simeq 24\pi \left(\frac{1}{80.4}\,\frac{\text{cm}}{5.06 \times 10^{13}}\right)^2 0.105 \simeq 5 \times 10^{-31}\text{cm}^2 \tag{16}$$

For comparison, Fig. 1 shows the cross sections for neutrino interactions on electron targets as function of neutrino energy.

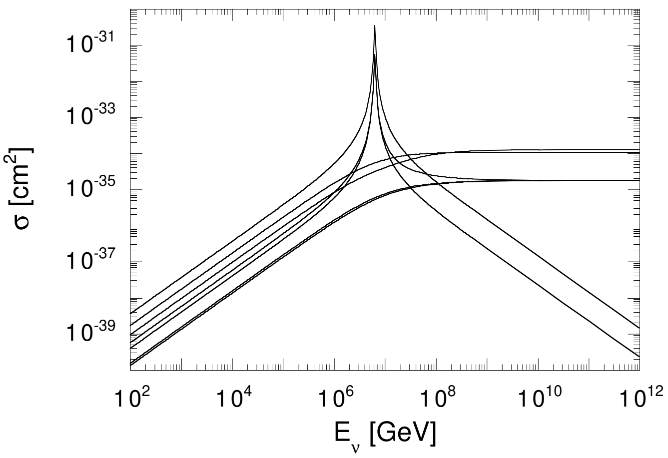


Figure 1: At low energies, from largest to smallest cross section, the processes are (i) $\bar{\nu}_e e \to$ hadrons, (ii) $\nu_\mu e \to \mu \nu_e$, (iii) $\nu_e e \to \nu_e e$, (iv) $\bar{\nu}_e e \to \bar{\nu}_\mu \mu$, (v) $\bar{\nu}_e e \to \bar{\nu}_e e$, (vi) $\nu_\mu e \to \nu_\mu e$, (vii) $\bar{\nu}_\mu e \to \bar{\nu}_\mu e$.

d.) The neutrino state ν_e created in beta-decay is a superposition of mass eigenstates ν_i,

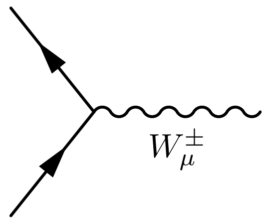
$$\nu_e(0) = U_{ei}\nu_i = U_{e1}\nu_1(0) + U_{e2}\nu_2(0) + U_{e3}\nu_3(0)$$

The phase of the mass eigenstates will evolve as

$$\nu_i(x) = \nu_i(0)\exp(\mathrm{i}p_i x) \simeq \nu_i(0)\mathrm{e}^{\mathrm{i}\phi}\exp[-\mathrm{i}Ex/(2m_i^2)]$$

The last term implies that the states devlop an oscillating phase difference as they propagate. At detection, we have to split $\nu_i(L)$ into flavor states, $\nu_i(L) = U_{i\alpha}^*\nu_\alpha$. Then $|\langle\nu_\alpha|\,\nu_i(L)\rangle|^2$ gives the probability to observe flavour alpha. As a result, the originally 100% pure ν_e becomes a superposition of all three flavors. In this case, the importance of the Glashow resonance would be thereby reduced.

Feynman rules and useful formulas



$$-\frac{\mathrm{i}g}{2\sqrt{2}}\gamma^{\mu}(1-\gamma^5)$$

The gamma matrices form a Clifford algebra,

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}\,. \tag{17}$$

The matrix

$$\gamma^5 \equiv \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{18}$$

satisfies $(\gamma^5)^{\dagger} = \gamma^5$, $(\gamma^5)^2 = 1$, and $\{\gamma^{\mu},\gamma^5\} = 0$.

$$\overline{\Gamma} \equiv \gamma^0\Gamma^{\dagger}\gamma^0\,. \tag{19}$$

The trace of an odd number of γ^{μ} matrices vanishes, as well as

$$\mathrm{tr}[\gamma^5] = \mathrm{tr}[\gamma^{\mu}\gamma^5] = \mathrm{tr}[\gamma^{\mu}\gamma^{\nu}\gamma^5] = 0\,. \tag{20}$$

Non-zero traces are

$$\mathrm{tr}[\gamma^{\mu}\gamma^{\nu}\ ] = 4\eta^{\mu\nu} \quad \text{and} \quad \mathrm{tr}[\not{a}\not{b}] = 4a\cdot b \tag{21}$$

$$\mathrm{tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(a\cdot b)\,(c\cdot d) - 4(a\cdot c)\,(b\cdot d) + 4(a\cdot d)\,(b\cdot c)] \tag{22}$$

$$\mathrm{tr}\left[\gamma^5\gamma_{\mu}\gamma_{\nu}\gamma_{\alpha}\gamma_{\beta}\right] = 4\mathrm{i}\varepsilon_{\mu\nu\alpha\beta} \tag{23}$$

Useful are also $\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b$, $\not{a}\not{a} = a^2$ and the following contractions,

$$\gamma^{\mu}\gamma_{\mu} = 4\,, \qquad \gamma^{\mu}\not{a}\gamma_{\mu} = -2\not{a}\,, \qquad \gamma^{\mu}\not{a}\not{b}\gamma_{\mu} = 4a\cdot b\,, \qquad \gamma^{\mu}\not{a}\not{b}\not{c}\gamma_{\mu} = -2\not{c}\not{b}\not{a}\,. \tag{24}$$

Completeness relations

$$\sum_s u_a(p,s)\bar{u}_b(p,s) = (\not{p}+m)_{ab}\ , \tag{25}$$

$$\sum_s v_a(p,s)\bar{v}_b(p,s) = (\not{p}-m)_{ab}\ . \tag{26}$$

Decay rate

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2E_i}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,. \tag{27}$$

The two particle phase space $\mathrm{d}\Phi^{(2)}$ in the rest frame of the decaying particle is

$$\mathrm{d}\Phi^{(2)} = \frac{1}{16\pi^2}\,\frac{|\boldsymbol{p}'_{\mathrm{cms}}|}{M}\,\mathrm{d}\Omega\,, \tag{28}$$

Breit-Wigner formula

$$\sigma(12 \to R \to 34) = \frac{4\pi s}{p_{\rm cms}^2} \frac{(2s_r+1)}{(2s_1+1)(2s_2+1)} \frac{\Gamma(R\to 12)\Gamma(R\to 34)}{(s-M_R^2)^2+(M_R\Gamma_R)^2} \tag{29}$$

where Γ_R is the total decay width of the resonance with mass m_R and spin s_R, while s_1 and s_2 are the spins of the particles in the initial state.

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_\phi^2\phi^2 + \bar\psi(\mathrm{i}\partial\!\!\!/ - m_\psi)\psi$$

	mass	energy	1/length	1/time	temperature
GeV	1.78×10^{-24} g	1.60×10^{-3} erg	5.06×10^{13} cm^{-1}	1.52×10^{24} s^{-1}	1.16×10^{13}K