Eksamination in FY3403 Particle physics
Wednesday December 6, 2017
Solutions

1a) The total angular momentum is zero in the initial state. It is conserved and must be zero in the final state, where the only contribution to the total angular momentum is the orbital angular momentum of the two pions.

Addition of two isospins 1 can give isospin either 0, 1, or 2.

1b) There are $3 \times 3 = 9$ states, from which we can make six symmetric states:

$$|++\rangle\ , \qquad |00\rangle\ , \qquad |--\rangle\ , \qquad |+0\rangle + |0+\rangle\ , \qquad |+-\rangle + |-+\rangle\ , \qquad |0-\rangle + |-0\rangle\ ,$$

and three antisymmetric:

$$|+0\rangle - |0+\rangle\ , \qquad |+-\rangle - |-+\rangle\ , \qquad |0-\rangle - |-0\rangle\ .$$

When we add two isospins 1, each of the isospins 0, 1, 2 occurs once. There are $2I + 1$ states of isospin I, hence the number of states are 1, 3, 5, with $1 + 3 + 5 = 9$.

It is then easy to guess that the three antisymmetric states are the isospin 1 states, and that the isospin 0 and 2 states are symmetric. The table of Clebsch–Gordan coefficients confirms this.

1c) They can not have isospin 1 because they are bosons and must have a symmetric wave function. The spatial part of the wave function is symmetric because $\ell = 0$. Hence the isospin part of the wave function must be symmetric, excluding isospin 1.

1d) The isospin of the neutral K meson is $I = 1/2$. Hence if the isospin change is $\Delta I = \pm 1/2$, the isospin in the final state of two π mesons must be eiter 0 or 1. Since $I = 1$ is excluded, the only remaining possibility is $I = 0$.

From the table of Clebsch–Gordan coefficients we see that the isospin zero state is

$$|I = 0, I_3 = 0\rangle = \frac{1}{\sqrt{3}} \left(|+-\rangle - |00\rangle + |-+\rangle \right)\ .$$

If we have two detectors, there are then three possibilities having equal probabilities:

1) π^+ in detector 1, π^- in detector 2;

2) π^0 in detector 1, π^0 in detector 2;

3) π^- in detector 1, π^+ in detector 2.

Hence, in total over (for example) 3000 decays there are in total 6000 π mesons, about 2000 each of π^+, π^0, and π^-.

2a) The neutral K mesons are pseudoscalars, transforming as follows under charge conjugation C and parity P (this is the P transformation of particles at rest, P reverses the momentum of a moving particle):

$$C|K^0\rangle = |\overline{K}^0\rangle\ , \qquad C|\overline{K}^0\rangle = |K^0\rangle\ , \qquad P|K^0\rangle = -|K^0\rangle\ , \qquad P|\overline{K}^0\rangle = -|\overline{K}^0\rangle\ .$$

Hence

$$CP|K^0\rangle = -|\overline{K}^0\rangle\ , \qquad CP|\overline{K}^0\rangle = -|K^0\rangle\ ,$$

and the states $|K_1\rangle$ and $|K_2\rangle$ are eigenstates of the CP operator with eigenvalues ± 1:

$$CP\,|K_1\rangle = |K_1\rangle\ , \qquad CP\,|K_2\rangle = -|K_2\rangle\ .$$

2b) If CP is conserved, then K_1 and K_2 have to decay into states with $CP = +1$ and $CP = -1$, respectively. Note that C alone and P alone are very far from conserved in weak interactions.

A final state with two π mesons must have $CP = +1$, and the reasoning behind this conclusion is as follows.

– Charge conservation allows the two possibilities $\pi^+\pi^-$ and $\pi^0\pi^0$.

– The intrinsic parity is -1 for one π meson, $(-1)^2 = +1$ for two π mesons, $(-1)^3 = -1$ for three π mesons, and so on.

– Because the K mesons and the π mesons all have spin zero, and the total angular momentum is conserved. the $\pi\pi$ final states must have orbital angular momentum $\ell = 0$ in the centre of mass. The parity of a spatial wave function with orbital angular momentum $\ell = 0$ is $(-1)^\ell = +1$. Hence, the parity of the $\pi\pi$ final state is $P = (-1)^2(-1)^\ell = +1$. Note that we must have $(-1)^\ell = +1$ for $\pi^0\pi^0$ for a second good reason: they are identical bosons so that their wave function must be symmetric. For $\pi^+\pi^-$ the last argument does not apply.

– Charge conjugation transforms a state with one π meson as follows:

$$C\,|\pi^+\rangle = |\pi^-\rangle\ , \qquad C\,|\pi^0\rangle = |\pi^0\rangle\ , \qquad C\,|\pi^-\rangle = |\pi^+\rangle\ .$$

Note in particular that the intrinsic charge conjugation symmetry of π^0 is $+1$. Hence, $C\,|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle$ and $C\,|\pi^+\pi^-\rangle = |\pi^-\pi^+\rangle$.

– Every wave function can be written as a sum of wave functions that are products of a spatial wave function and an isospin wave function. The operation P affects only the spatial part of a product wave function, whereas C affects only the isospin part.

– Thus, in the state $\pi^+\pi^-$ the operation C interchanges the two particles in the isospin part of a product wave function, whereas P interchanges them in the spatial part. Since the π mesons are bosons, the total wave function must be symmetric, and we must have $CP = +1$ for the $\pi^+\pi^-$ state. As we have seen, the spatial part has the symmetry $(-1)^\ell = 1$ because $\ell = 0$. To make the total wave function symmetric we have to make also the isospin wave function symmetric.

– In conclusion, both $\pi\pi$ final states have $P = +1$ and $C = +1$, and hence $CP = +1$.

A final state with three π mesons must have $CP = -1$, because of the intrinsic parity $P = (-1)^3 = -1$. We reason as follows.

– Charge conservation allows the two possibilities $\pi^+\pi^-\pi^0$ and $\pi^0\pi^0\pi^0$.

– Assuming that all orbital angular momenta are zero, the spatial wave function is symmetric under the interchange of any two particles. Since the particles are bosons, the total wave function must be symmetric, hence the isospin wave function must also be symmetric. Charge conjugation interchanges the π^+ and π^- in the isospin wave function, for example $C\,|\pi^+\pi^-\pi^0\rangle = |\pi^-\pi^+\pi^0\rangle$, but the isospin wave function is symmetric and does not change.

– In conclusion, both $\pi\pi\pi$ final states have $P = -1$, $C = +1$, and $CP = -1$. There is one (small) loophole in this argument: there could be a nonzero orbital angular momentum between the π^+ and the π^- in the $\pi^+\pi^-\pi^0$ state, but this is unlikely because the kinetic energy $(m_K - 3m_\pi)c^2$ is low.

Assuming that CP is conserved we conclude that K_1 may decay to two π mesons, whereas K_2 has to decay to three π mesons. The mass of the neutral K meson is 497 MeV/c^2, whereas the sum of the three π masses is 405 or 414 MeV/c^2. Thus there is little phase space available for the decay to three π mesons, implying that this decay is much slower than the decay to two π mesons.

2c) Define $f = f(t) = \mathrm{e}^{-\left(\mathrm{i}m_S + \frac{\Gamma_S}{2}\right)t}$ and $g = g(t) = \mathrm{e}^{-\left(\mathrm{i}m_L + \frac{\Gamma_L}{2}\right)t}$, so that

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(f\,|K_1\rangle + g\,|K_2\rangle\right) = \frac{1}{2}\left((f+g)\,|K^0\rangle - (f-g)\,|\overline{K}^0\rangle\right) .$$

The unnormalized probability for K^0 is

$$\begin{aligned} q_1 &= |f+g|^2 = (f^* + g^*)(f+g) = |f|^2 + |g|^2 + f^*g + g^*f \\ &= \mathrm{e}^{-\Gamma_S t} + \mathrm{e}^{-\Gamma_L t} + 2\,\mathrm{e}^{-\overline{\Gamma}\,t}\cos(\Delta m\,t) \\ &= 2\,\mathrm{e}^{-\overline{\Gamma}\,t}\left(\cosh(\gamma t) + \cos(\Delta m\,t)\right) , \end{aligned}$$

when we define

$$\overline{\Gamma} = \frac{\Gamma_S + \Gamma_L}{2} , \qquad \gamma = \frac{\Gamma_S - \Gamma_L}{2} , \qquad \Delta m = m_L - m_S .$$

The unnormalized probability for $\overline{K}^0$ is

$$\begin{aligned} q_2 &= |f-g|^2 = (f^* - g^*)(f-g) = |f|^2 + |g|^2 - f^*g - g^*f \\ &= \mathrm{e}^{-\Gamma_S t} + \mathrm{e}^{-\Gamma_L t} - 2\,\mathrm{e}^{-\overline{\Gamma}\,t}\cos(\Delta m\,t) \\ &= 2\,\mathrm{e}^{-\overline{\Gamma}\,t}\left(\cosh(\gamma t) - \cos(\Delta m\,t)\right) . \end{aligned}$$

The normalized probability for K^0 is then

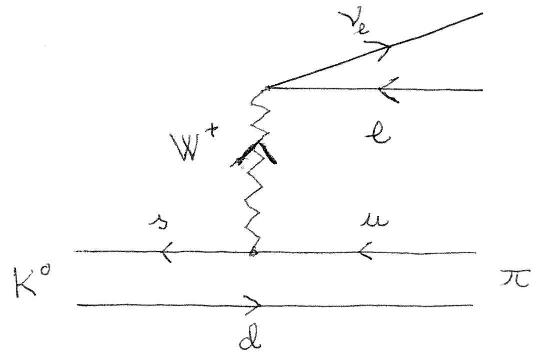
$$p_1 = \frac{q_1}{q_1 + q_2} = \frac{1}{2} + \frac{\cos(\Delta m\,t))}{2\cosh(\gamma t)} ,$$

whereas the normalized probability for $\overline{K}^0$ is

$$p_2 = \frac{q_2}{q_1 + q_2} = \frac{1}{2} - \frac{\cos(\Delta m\,t))}{2\cosh(\gamma t)} .$$

We see that $p_1(t) \to 1/2$ and $p_2(t) \to 1/2$ when $t \to \infty$. We could predict these limits without computing them, since the state, if it survives for a long time, will become more and more like the long lived K_2, which is 50 % K^0 and 50 % $\overline{K}^0$.

2d) The Feynman diagram shown here is the lowest order diagram for the decay $K^0 \to \pi^- e^+ \nu_e$. An $\overline{s}$ quark emits a W^+ and is turned into an $\overline{u}$ quark, thereby changing its strangeness by $\Delta S = -1$ (from $+1$ to 0) and its electric charge by $\Delta Q = -1$ (from $+1/3$ to $-2/3$). In the charge conjugated decay process $\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e$ all signs (and fermion arrows) are reversed. In both cases, $\Delta S = \Delta Q$.



2e) The semileptonic decays $K^0 \to \pi^- e^+ \nu_e$ and $\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e$ can be transformed into each other by a CP transformation. Hence, if CP invariance is exact the decay rates for these two decay modes have to be equal.

It follows that if we detect equal numbers of the decays $K^0 \to \pi^- e^+ \nu_e$ and $\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e$ in a given time interval, we may conclude that we started out with equal numbers of K^0 and $\overline{K}^0$ particles.

2f) With the assumptions we have made we have

$$\delta(t) = p_1(t) - p_2(t) = \frac{\cos(\Delta m\, t))}{\cosh(\gamma t)} .$$

This implies in particular that $\delta(t = 0) = 1$, and that $\delta(t) \to 0$ when $t \to \infty$, as we concluded in part 2c).

Maybe we should write $t - t_0$ instead of t, since we may not know very precisely, in an experiment, the time t_0 when the particle is produced.

2g) Our theoretical formula for $\delta(t)$ is independent of the sign of Δm. It has been shown experimentally, in other ways, that $\Delta m > 0$. Here we only get information about $|\Delta m|$.

The most accurate estimate of Δm, at least by eye from the figure, is probably from the zeros of the asymmetry $\delta(t)$. According to our theoretical formula we have $\delta(t) = 0$ first for $\Delta m\,(t - t_0) = \pi/2$ and next for $\Delta m\,(t - t_0) = 3\pi/2$. It seems a reasonable precaution not to trust too much the zero point of the time axis. Therefore we use the two visible zeros of $\delta(t)$, which we estimate to lie at $t_1 = 0.27$ ns and at $t_2 = 0.94$ ns, taking into account the fact which is obvious from the figure that the limiting value of $\delta(t)$ when $t \to \infty$ is not 0. Thus we should have that

$$3(t_1 - t_0) = t_2 - t_0 , \qquad t_0 = \frac{3t_1 - t_2}{2} = -0.065 \text{ ns} ,$$

and

$$\Delta m\,(t_2 - t_1) = \Delta m \times 0.67 \text{ ns} = \pi \;.$$

Remembering that we have set $\hbar = 1$ and $c = 1$, we find that

$$\begin{aligned}\Delta m &= \frac{\pi}{0.67 \text{ ns}} = 4.7 \times 10^9/\text{s} = 4.7 \times 10^9 \;(\hbar/\text{s})/c^2 \\ &= 4.7 \times 10^9 \times 6.58 \times 10^{-22} \text{ MeV}/c^2 = 3.1 \times 10^{-12} \text{ MeV}/c^2 \;,\end{aligned}$$

and hence

$$\frac{\Delta m}{m_{K^0}} = \frac{3.1 \times 10^{-12} \text{ MeV}/c^2}{497.61 \text{ MeV}/c^2} = 6.2 \times 10^{-15} \;.$$

The official numbers from the Particle Data Group are $\Delta m = 3.483 \times 10^{-12}$ MeV$/c^2$ and $\Delta m/m_K = 7.000 \times 10^{-15}$.

One way (there may be others?) to estimate the rate γ is to look at the negative minimum value of $\delta(t)$, which is around -0.076 at time $t_3 = 0.43$ ns, or perhaps rather -0.080, since the limiting value for $\delta(t)$ as $t \to \infty$ is not zero but around 0.004. Hence, for $t_3 = 0.43$ ns we should have

$$\Delta m\,(t_3 - t_0) = \Delta m\,(t_1 - t_0)\,\frac{t_3 - t_0}{t_1 - t_0} = \frac{\pi}{2}\,\frac{0.43 + 0.065}{0.27 + 0.065} = 0.739\,\pi = 2.32 \;.$$

This gives the minimum value

$$\delta(t_3) = \frac{\cos(\Delta m\,(t_3 - t_0))}{\cosh(\gamma(t_3 - t_0))} \approx 2\cos(0.739\,\pi)\,\mathrm{e}^{-\gamma(t_3 - t_0)} = -1.3645\,\mathrm{e}^{-\gamma\,0.495 \text{ ns}} = -0.080 \;.$$

And it gives

$$\gamma = \frac{\ln 1.3645 - \ln 0.080}{0.495 \text{ ns}} = 5.73/\text{ns} = 1.22\;\Delta m \;.$$

To determine a more precise value of γ giving a minimum value of -0.080, if we have Maple available, we may simply plot $\delta(t)$, with $\Delta m = 4.7/\text{ns}$, in the interval $0.4 \text{ ns} < t < 0.5$ ns and try different values of γ. With $\gamma = 5.73/\text{ns}$ we get the wanted minimum value of -0.080 at $t = 0.42$ ns.

Since we happen to know that $\Gamma_L \ll \Gamma_S$ we may take

$$\Gamma_S = 2\gamma + \Gamma_L \approx 2\gamma = 1.146 \times 10^{10}/\text{s} = \frac{1}{8.7 \times 10^{-11} \text{ s}} \;.$$

According to these measurements and calculations the mean lifetime of K_S should be 8.7×10^{-11} s (the Particle Data Group says 8.96×10^{-11} s).

The fact that Δm and γ are very nearly equal (and not ten orders of magnitude different as they might have been) calls for an explanation. I will not try to explain this remarkable coincidence.

2h) The figure shows that the limiting value of $\delta(t)$ for t large is around 0.004 (the official experimental result is 0.00332 ± 0.00006). Since we calculated a limiting value 0 under the assumption of CP invariance, we have here an experimental proof that the CP invariance is broken. And the size of the breaking is about 0.4 %.