FY3403 Exam 2021: Solutions

Question 1 (18 points)

(a) A fixed-target collision has:

$$\begin{aligned} s = (p_A + p_B)^2 &= = p_A^2 + p_B^2 + 2p_A p_B \\ &= M_A^2 + M_B^2 + 2E_A M_B \end{aligned} \tag{1}$$

So

$$\sqrt{s} \approx \sqrt{2\,E\,m_P}$$

So for cosmic rays

$$\sqrt{s} = \sqrt{2 \times 10^{11} \times 0.938} \sim 433\,\text{TeV}$$

For a collider :

$$\begin{aligned} s = (p_A + p_B)^2 &= = p_A^2 + p_B^2 + 2p_A p_B \\ &= M_A^2 + M_B^2 + 2E_A E_B + 2|\mathbf{p}_A||\mathbf{p}_B| \\ \Rightarrow \sqrt{s} &\approx \sqrt{4E_A E_B} \end{aligned}$$

Plugging in the numbers :

$$\sqrt{s} = \sqrt{4 \times 7 \times 7} = 14\,\text{TeV}$$

So the ratio is $433/14 \approx 31$

[**8**]

(b) We have :

$$\begin{aligned} p_C &= p_A + p_B \\ p_C^2 + p_A^2 - 2p_C p_A &= p_B^2 \\ \Rightarrow m_C^2 + m_A^2 - 2(E_C E_A - \mathbf{p}_C \cdot \mathbf{p}_A) &= m_B^2 \end{aligned}$$

In the parent particle's rest frame, $\mathbf{p}_C = 0$ and $E_C = m_C$ so :

$$\begin{aligned} m_C^2 + m_A^2 - 2m_C E_A &= m_B^2 \\ \Rightarrow E_A &= \frac{m_C^2 + m_A^2 - m_B^2}{2m_C} \\ E_B &= \frac{m_C^2 + m_B^2 - m_A^2}{2m_C}\,. \end{aligned}$$

[**5**]

(c) In the rest frame of C :

$$
\begin{aligned}
m_C = E_A + E_B &= \sqrt{m_A^2 + \mathbf{p}^2} + \sqrt{m_B^2 + \mathbf{p}^2} \\
m_B^2 + \mathbf{p}^2 &= \left(m_C - \sqrt{m_A^2 + \mathbf{p}^2}\right)^2 \\
m_B^2 + \mathbf{p}^2 &= m_C^2 + m_A^2 + \mathbf{p}^2 - 2m_C\sqrt{m_A^2 + \mathbf{p}^2} \\
4m_C^2(m_A^2 + \mathbf{p}^2) &= \left(m_C^2 + m_A^2 - m_B^2\right)^2 \\
4m_A^2 m_C^2 + 4m_C^2\mathbf{p}^2 &= m_c^4 + m_A^4 + m_B^4 + 2m_A^2 m_C^2 - 2m_B^2 m_C^2 - 2m_A^2 m_B^2 \\
|\mathbf{p}| &= \left[m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_B^2 m_C^2 - 2m_A^2 m_C^2\right]^{1/2} / 2m_C
\end{aligned}
$$

[5]

Question 2 (10 points)

(a) The decay is $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. The coupling at each vertex is proportional to g and the decay rate is proportional to the amplitude squared so $\Gamma \sim g^4 \sim G_F^2$ [1]. The coupling of the W is the same at the muon vertex as it is in the electron vertex due to lepton Universality.

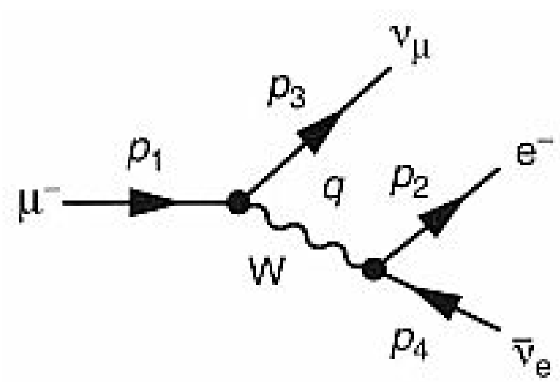

Figure 1: Muon decay Feynman diagram.

(b)

$$
\begin{aligned}
\Gamma = \frac{1}{\tau} &= \frac{G_F^2 m_\mu^5}{192\pi^3} \\
&= \frac{(1.16637 \times 10^{-5})^2 \cdot (0.105658367)^5}{192\pi^3} \\
&= 3.00913246 \times 10^{-19}\,\text{GeV} \\
\Rightarrow \tau &= \frac{\hbar}{\Gamma} \\
&= 2.19 \times 10^{-6}\,\text{s}\,.
\end{aligned}
$$

(c) Therefore for a τ particle :

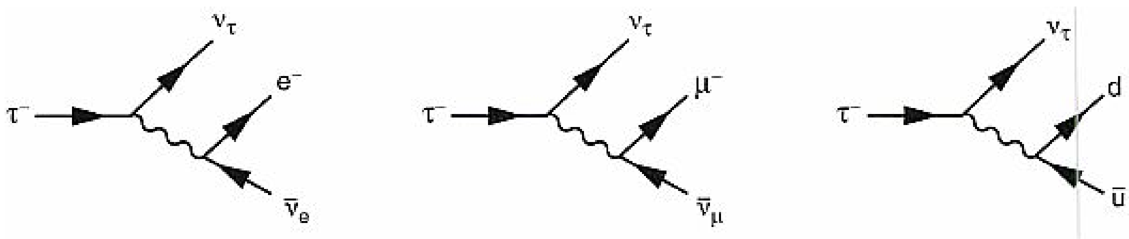

Figure 2: Tau decay Feynman diagram. Only the decay to an electron is required.

$$\begin{aligned}\tau(\tau^- \to e^- \bar{\nu}_e \nu_\tau) &= 2.19 \times 10^{-6} \cdot \left(\frac{m_\mu}{m_\tau}\right)^5 \cdot 17.85\% \\ &= 2.903 \times 10^{-13}\,\mathrm{s}\,.\end{aligned}$$

Question 3 (15 points)

(a) Figure here (or see e.g. Fig 4.12 of Griffiths).

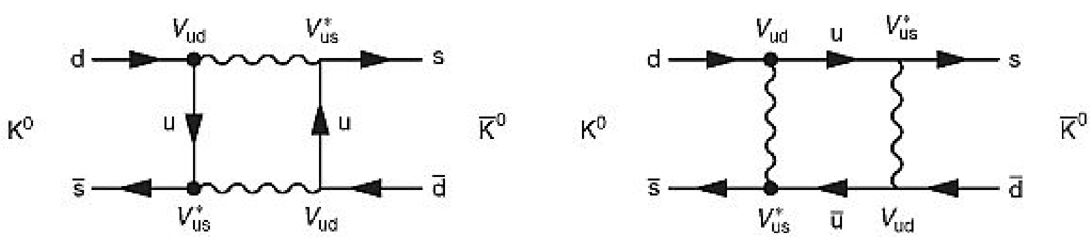

Figure 3: Neutral kaon oscillation Feynman diagram

(b) The K^0 and $\bar{K}^0$ belong to a $J^{PC} = 0^{-+}$ multiplet so

$$CP|K^0\rangle = -|\bar{K}^0\rangle, \quad CP|\bar{K}^0\rangle = -|K^0\rangle.$$

The CP eigenstates K_1 and K_2 can then be constructed as

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K^0}\rangle\right) \quad CP|K_1\rangle = +|K_1\rangle$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K^0}\rangle\right) \quad CP|K_2\rangle = -|K_2\rangle$$

If CP violation is neglected, the states K_S and K_L decay only via $K_S \to \pi\pi$ and $K_L \to \pi\pi\pi$. The $\pi\pi$ system has CP = +1 and the $\pi\pi\pi$ system has CP = -1, and we can therefore identify

$$|K_S\rangle = |K_1\rangle$$

$$|K_L\rangle = |K_2\rangle.$$

(c) The K_0 is a superposition of K_1 and K_2 (or K_L and K_S). If we start with a beam of K^0s

$$|K_0\rangle = \frac{1}{\sqrt{2}}\left(|K_S\rangle - |K_L\rangle\right).$$

The K_S will quickly decay away, so close to the production point we'll observed mostly decays to two pions whereas sufficiently far away we'll detect decays to three pions.

(d) If a small degree of CP violation occurs the K_L is in a state of the form

$$|K_L\rangle = \frac{1}{\sqrt{1+\epsilon^2}}\left(|K^2\rangle - \epsilon|K_1\rangle\right)$$

At large distances, where we would have otherwise expected only decays to three pions we will in this case see a small fraction of decays to two pions.

Question 4 (23 points)

(a) Figure (Fig in 10.4 of Griffiths or Fig. 11.9 Thomson).

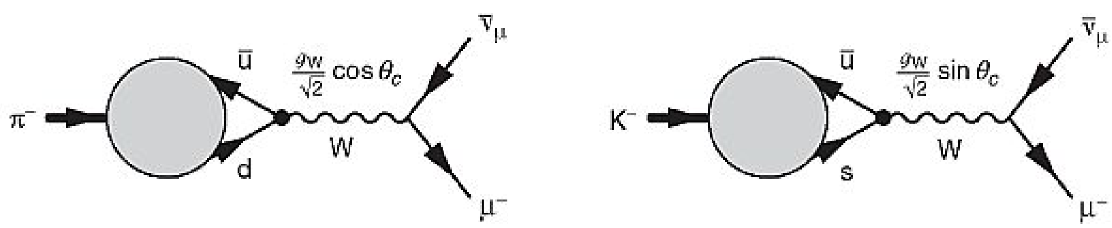

Figure 4: Pion decay Feynman diagram

(b) The π^- is a spin-0 meson, so in its rest frame we have equal and opposite muon and neutrino momenta and they must have equal and opposite helicities. The antineutrino is an antiparticle and so it is produced in the right handed (RH) chiral state by the W-boson. As an effectively massless particle, the neutrino's RH chiral state is coincident with a RH helicity state and therefore, to conserve total spin the muon must also be in a RH helicity state.

(c) The time-like component of the four-vector current is

$$j_l^0 = \bar{u}(p_3)\frac{1}{2}\gamma^0(1-\gamma^5)\upsilon(p_4)$$

But, $\bar{u}\gamma^0 = u^\dagger\gamma^0\gamma^0 = u^\dagger$, so,

$$j_l^0 = u^\dagger(p_3)\frac{1}{2}(1-\gamma^5)\upsilon(p_4)$$

For the neutrino since $m \approx 0$ the helicity states are essentially equivalent to the chiral states. Therefore,

$$\frac{1}{2}(1-\gamma^5)\upsilon(p_4) = \upsilon_\uparrow(p_4)$$

.

From the discussion in (b) the helicity of the muon must be the same. So the time-like component reduces to

$$j_l^0 = u_\uparrow^\dagger(p_3)\upsilon_\uparrow(p_4)$$

The helicity spinors are thus

$$u_\uparrow(p_3) = \sqrt{E_l + m_l}\begin{pmatrix}1\\0\\\alpha\\0\end{pmatrix} \quad \upsilon_\uparrow(p_4) = \sqrt{|\mathbf{p}|}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \tag{2}$$

And therefore, $j_l^0 = u_\uparrow^\dagger(p_3)\upsilon_\uparrow(p_4) = u_\uparrow(p_3)\upsilon_\uparrow(p_4) = \sqrt{E_l + m_l}\sqrt{p}(1-\alpha)$.
Substituting in α,

$$j_l^0 = \sqrt{E_l + m_l}\sqrt{p}\left(1 - \frac{|p|}{E_l + m_l}\right) = \sqrt{p}\frac{E_l + m_l - |p|}{\sqrt{E_l + m_l}}.$$

(d) In general $\mathcal{M}^2 \propto (j_l^\mu)^2$. The question further tells us that only the time-like component contributes, so $\mathcal{M}^2 \propto (j_l^0)^2$.
From (c)

$$\begin{aligned} j_l^2 &= \frac{p(E+m-p)^2}{E+m} \\ &= \frac{p(E^2+m^2+p^2+2Em-2Ep-2mp)^2}{E+m} \\ &= \frac{p(2E^2+2Em-2Ep-2mp)^2}{E+m} \\ &= \frac{2p(E^2+Em-Ep-mp)^2}{E+m} \\ &= \frac{2p(E-p)(E+m)}{E+m} \\ &= 2p(E-p) \end{aligned} \tag{3}$$

Using the supplied two-body decay rate together with the result just proved

$$\frac{\Gamma(\pi^+ \to e^+\bar{\nu}_e)}{\Gamma(\pi^+ \to \mu^+\bar{\nu}_\mu)} = \left(\frac{p_e}{p_\mu}\right)\frac{p_e(E_e - m_e)}{p_\mu(E_\mu - p_\mu)}$$

Using the numerical values provided the ratio $\approx 1.2 \times 10^{-4}$ is obtained.

e. Some essential points: (i) Mention of Wu's experiment. (ii) Cobalt spins are aligned in the experimental apparatus. (iii) Preferred electron emission in one direction with respect to the spin of the Cobalt nuclei (antiparallel to the spin axis). (iv) The asymmetry (i.e. the fact that there is a preferred direction of the electron momentum) is unambiguous evidence of parity violation in weak interactions.

[Total Marks = 67]

END OF PAPER