NTNU – Trondheim
Norwegian University of
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Exam FY3452 Gravitation and Cosmology Spring 2016

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09.00-13.00

Permitted examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. Read carefully. Good luck! Bonne chance! Viel Glück! Lykke til! In the problems, we use $c = G = 1$.

Problem 1

a) Consider two inertial frames S and S', where S' moves along the x-axis with velocity v. Write down the transformation that expresses t' x', y', and z' as functions of t, x, y, and z.

A disc of radius r is rotating counterclockwise with angular speed β. Its center is located at the origin in the xy-plane. A light source on the edge of the disc is emitting radiation at a frequency ω' in the rest frame of the source. When the source crosses the y-axis in the lower half-plane, it emits radiation in the y' direction, where S' denotes the instantaneous rest frame.

b) Find the frequency ω and the components of the wavevector $\mathbf{k}$ in the lab frame S in terms of the corresponding quantities ω' and $\mathbf{k}'$ in S'.

c) Find the speed $v = \beta r$ such that the angle between the radiation and x-axis in S is $\frac{1}{4}\pi$.

Problem 2

Consider a two-dimensional space with the line element

$$ds^2 \;=\; dr^2 + f(r)d\phi^2\ , \tag{1}$$

where r and ϕ are coordinates with range $0 \le r < \infty$ and $0 \le \phi \le 2\pi$, and $f(r)$ is a smooth real function.

a) The only nonzero Christoffel symbols are $\Gamma^r_{\phi\phi}$ and $\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r}$. Calculate the nonzero Christoffel symbols.

b) The only nonzero components of the Ricci tensor are R_{rr} and $R_{\phi\phi}$. Calculate the nonzero components of the Ricci tensor.

c) Use this to show that the Ricci scalar R can be written as

$$R \;=\; \frac{1}{2}\frac{[f'(r)]^2}{f^2(r)} - \frac{f''(r)}{f(r)}\ . \tag{2}$$

d) Finally assume that $f(r)$ is of the form

$$f(r) \;=\; r^n\ , \tag{3}$$

where n is nonnegative integer. For which values of n is the space flat? For which values is the space Euclidean?

Problem 3

In this problem, we are going to study some of the properties of an electrically charge and spherically symmetric black hole. The metric was found by Reissner and Nordstrom and reads

$$ds^2 \;=\; -\left(1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2\ , \tag{4}$$

where m is the mass and $Q = \varepsilon^2$ is the electric charge of the black hole.

a) $r = 0$ is a coordinate singularity (and a physical one as well). Show that the other coordinate singularities are given by

$$r_\pm = m \pm \sqrt{m^2 - \varepsilon^2} \, . \tag{5}$$

We can use $r_\pm$ to divide r into the three different regions according to

$$\begin{aligned} \text{I} : \quad & 0 < r < r_- \, , \\ \text{II} : \quad & r_- < r < r_+ \, , \\ \text{III} : \quad & r_+ < r \, . \end{aligned} \tag{6}$$

b) Using a clever coordinate transformation, the line element can be written in the form

$$ds^2 = -(1-f)d\bar{t}^2 + 2f d\bar{t}dr + (1+f)dr^2 + r^2 d\Omega^2 \, , \tag{7}$$

where $f = \frac{2m}{r} - \frac{\varepsilon^2}{r^2}$. Show that a family of radial null geodesics are given by

$$\bar{t} + r = \text{constant} \, . \tag{8}$$

Is this family of geodesics incoming or outgoing? Draw the geodesics in an $(\bar{t}, r)$-diagram.

c) Show that there is another family of radial null geodesics given by

$$\frac{d\bar{t}}{dr} = \frac{1+f}{1-f} \, . \tag{9}$$

Fig. 1 shows $1-f$ and $1+f$ as functions of r. Use this to sketch the geodesics that are the solutions to Eq. (9) in a $(\bar{t}, r)$-diagram.

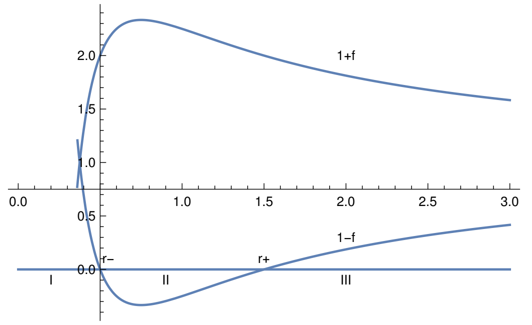

Figure 1: Plot of $1+f$ and $1-f$ as functions of r. The zeros of $1-f$ are at $r_\pm$.

d) Show or explain that $r = r_+$ is an event horizon.

e) Once the particle is in region I, is it bound to fall into the singularity at $r = 0$?

f) We finally specialize to the case where $\varepsilon^2 = \frac{3}{4}m^2$. What are the corresponding values of r_+ and r_-? Calculate the proper time $\Delta\tau$ it takes for a particle to travel from r_+ to r_- starting at rest.

Useful formulas

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right], \tag{10}$$

$$R_{\alpha\beta} = \partial_\gamma \Gamma^{\gamma}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\delta}_{\beta\gamma}\Gamma^{\gamma}_{\alpha\delta}, \tag{11}$$

$$R = g^{\alpha\beta}R_{\alpha\beta}. \tag{12}$$