NTNU – Trondheim
Norwegian University of
Science and Technology

Exam FY3452 Gravitation and Cosmology fall 2016

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09.00-13.00

Permitted examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. Read carefully. Good luck! Bonne chance! Viel Glück! Veel succes! Lykke til!

Problem 1

Consider the standard situation where an inertial frame S' moves along the positive x-axis with speed v relative to another inertial frame S.

a) Show the relation between the acceleration a'_x in S' and a_x in S

$$a'_x = \frac{1}{\gamma}\frac{a_x}{(1-\frac{vV_x}{c^2})^2} + \frac{1}{\gamma}\frac{V_x - v}{(1-\frac{vV_x}{c^2})^3}\frac{va_x}{c^2} \tag{1}$$

Show that if S' is the instantaneous rest frame of a particle moving along the x-axis in S, a'_x reduces to

$$a'_x = \gamma^3 a_x \,. \tag{2}$$

b) Assume that the acceleration of the particle moving along the x-axis is constant in its instantaneous rest frame and equal to $a'_x = g$. Show that $V_x(t) = \frac{dx}{dt}$ is

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}} \,, \tag{3}$$

if the initial condition is $V_x(0) = 0$. What is the limiting velocity V_{lim} of $V_x(t)$ as $t \to \infty$?

c) The time t in S can be expressed as a function of the proper time τ of the particle. Show that

$$t(\tau) = \frac{c}{g}\sinh(\tfrac{g}{c}\tau) \,, \tag{4}$$

if the initial condition is $t(0) = 0$.

d) The position x in S can be expressed as a function of the proper time τ of the particle. Show that

$$x(\tau) = \frac{c^2}{g}\left[\cosh(\tfrac{g}{c}\tau) - 1\right] \,, \tag{5}$$

if the initial condition is $x(0) = 0$.

e) The functions $t(\tau)$ and $x(\tau)$ are the time and position of the origin of S' in S. We next consider an arbitrary point in spacetime, whose coordinates in S' are x' and $t' = \tau$. The coordinates of this point in S are given by

$$t = \left[\frac{c}{g} + \frac{x'}{c}\right]\sinh(\tfrac{g}{c}\tau) \,, \tag{6}$$

$$x = \frac{c^2}{g}\left[\cosh(\tfrac{g}{c}\tau) - 1\right] + x'\cosh(\tfrac{g}{c}\tau) \,. \tag{7}$$

Show that the metric can be written as

$$ds^2 = -c^2 dt'^2\left(1 + \frac{gx'}{c^2}\right)^2 + dx'^2 + dy'^2 + dz'^2 \,. \tag{8}$$

f) Explain why $\xi = (1, 0, 0, 0)$ is a Killing vector and find the associated conserved quantity.

g) Calculate the redshift of a photon that is emitted at $x' = h$ and absorbed at $x' = 0$. Explain the result.

Problem 2

The two Friedman equations are given by

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi\rho + \Lambda , \quad (9)$$

$$\frac{2\ddot{a}a + \dot{a}^2 + k}{a^2} = -8\pi p + \Lambda , \quad (10)$$

where $a(t)$ is the scale factor, $k = 0, \pm 1$ is the spatial curvature, ρ is the energy density of matter and radiation, p is the pressure, and $\Lambda > 0$ is the cosmological constant.

a) In Einstein's static model for the universe, there is no radiation present ($\rho = \rho_m$) and the pressure vanishes. Moreover the spatial curvature is positive, $k = +1$. Show that

$$\ddot{a} = -\frac{4\pi}{3}a\rho_m + \frac{1}{3}a\Lambda . \quad (11)$$

b) For a given value of Λ, there is a critical value of ρ_m, ρ_m^c, such that a is time independent. Find the value of ρ_m^c in terms of Λ. Find the corresponding value of $a = a_c$ in terms of Λ.

c) We will next study the stability of the static universe. We consider a small perturbation $\delta\rho_m$ of the density around ρ_m^c and write $\rho_m = \rho_m^c + \delta\rho_m$. we can then write $a = a + \delta a$, where δa is the corresponding change in the scale factor. $\delta\rho_m$ and δa are time dependent. Use the Friedman equations to show that δa satisfies the second-order differential equation

$$\frac{d^2\delta a}{dt^2} = B\delta a , \quad (12)$$

where B is a constant. Calculate B. Use this result to determine whether Einstein's static universe is stable or unstable. (Help: Even if you cannot find B, you can still say something about the stability).

Useful formulas

$$x' = \gamma(x - vt) \, , \tag{13}$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \, , \tag{14}$$

$$V'_x = \frac{V_x - v}{1 - \frac{vV_x}{c^2}} \, . \tag{15}$$