NTNU – Trondheim
Norwegian University of
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Exam FY3452 Gravitation and Cosmology summer 2018

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09.00-13.00

Permitted examination support material:
Rottmann: Matematisk Formelsamling
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Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. We use units such that $c = G = 1$. The metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Read carefully. Good luck!

Problem 1

Consider a four-dimensional spacetime whose geometry is described by the line element

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \ , \tag{1}$$

where M is a positive constant, and r, θ, and ϕ are spherical coordinates.

a) Classify the singularities in the geometry described by the line element (1). No proof is required.

b) Find two symmetries of the line element (1). Write down the corresponding Killing vectors ξ and η, and the conserved quantities e and l. Give a physical interpretation of e and l.

c) A massive particle is moving in the spacetime described by the line element (1). Use the results in **b)** to derive an equation of the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \;=\; \frac{1}{2}\left(e^2-1\right) \;, \tag{2}$$

where $V_{\text{eff}}(r)$ is a socalled effective potential.

d) A massive particle starts at rest at $r=\infty$ falling radially inwards. Show that the smallest radius is $r_{\text{min}} = \frac{1}{2}M$. Calculate the proper time $\Delta\tau$ it takes for the particle to travel from $r = M$ to $r_{\text{min}} = \frac{1}{2}M$.

e) We introduce a new time coordinate $\tilde{t}$ via

$$d\tilde{t} \;=\; dt - dr + \frac{dr}{\left(1-\frac{M}{r}\right)^2} \;. \tag{3}$$

Find the differential equations for the light-like curves in the spacetime specified by the line element (1). Sketch the worldlines of light in the $(r,\tilde{t})$-plane. Is this the geometry of a black hole?

f) We introduce a new variable v via

$$dv \;=\; d\tilde{t} + dr \;. \tag{4}$$

Express the line element in terms of dv, dr, and $d\Omega^2$.

g) Kepler's third law is $\Omega^2 = M/r^3$, where $\Omega = \frac{d\phi}{dt}$. Find the corresponding result in the spacetime described by the line element (1).

h) Calculate the gravitational redshift of a photon emitted in the radial direction by a stationary observer at $r_A > M$ and received at $r = \infty$. The emitted photon has frequency ω_A and the received photon has frequency ω_∞.

Problem 2

Consider the two-dimensional spacetime with the line element

$$ds^2 \;=\; -X^2 dT^2 + dX^2 \,. \tag{5}$$

a) Calculate the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$.

b) Calculate the diagonal components of the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R.

c) Is the the spacetime with the line element (5) flat? Prove your claim!

Problem 3

a) Write down the equation that defines parallel transport of a vector field A^α along a curve whose coordinates are $x^\alpha(\sigma)$, where σ is a parameter. Use this equation to give a definition of geodesics in terms of parallel transport.

b) The covariant derivative of a covariant tensor of rank two $t_{\alpha\beta}$ is defined by

$$\nabla_\gamma t_{\alpha\beta} \;=\; \partial_\gamma t_{\alpha\beta} - \Gamma^\delta_{\alpha\gamma} t_{\beta\delta} - \Gamma^\delta_{\beta\gamma} t_{\alpha\delta} \,. \tag{6}$$

Calculate the covariant derivative of the metric tensor $g_{\alpha\beta}$.

Problem 4

The line element for an isotropic and homogeneous universe is

$$ds^2 \;=\; -dt^2 + a^2(t)\left[d\chi^2 + \left\{ \begin{array}{l} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{array} \right\} (d\theta^2 + \sin^2\theta\, d\phi^2) \right] \,, \quad \left\{ \begin{array}{l} k = 1 \\ k = 0 \\ k = -1 \end{array} \right\} .$$

a) Explain briefly the terms isotropic and homogeneous. The spatial geometry of an isotropic and homogeneous universe is given by the value of k. Describe briefly the different geometries given by $k = 0, \pm 1$.

b) What is $a(t)$? What is the time dependence of $a(t)$ in a universe that has only a positive constant vacuum energy density Λ?

Useful formulas

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left[\frac{\partial g_{\alpha\delta}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\delta}} \right] , \tag{7}$$

$$R_{\alpha\beta} = \partial_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \partial_{\beta} \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\beta} \Gamma^{\delta}_{\gamma\delta} - \Gamma^{\delta}_{\beta\gamma} \Gamma^{\gamma}_{\alpha\delta} . \tag{8}$$