NTNU – Trondheim
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Exam FY3452 Gravitation and Cosmology fall 2018

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09.00-13.00

Permitted examination support material:
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of five pages. Read carefully. Good luck! Bonne chance! Viel Glück! Veel succes! Lykke til!

Problem 1

a) The spaceship NTNU2018 is moving along the x-axis in an inertial frame S. The initial conditions are $x(t=0) = V(t=0) = 0$. The clocks in the inertial frame and onboard the spaceship are such that $\tau = 0$ when $t = 0$. The acceleration in the instantenous rest frame is constant and equals g. It can be shown that

$$\frac{dt}{d\tau} = \cosh(\tfrac{g}{c}\tau) \ . \tag{1}$$

Use this to find $\frac{dx}{d\tau}$, $t(\tau)$, and $x(\tau)$. Show that the motion is hyperbolic.

b) At time $t = t_0 > 0$, a light signal is sent from the origin along the x-axis. Draw a spacetime diagram and the world lines of the spaceship and the photon. Show that a light signal sent later than $t_0 = \frac{c}{g}$ from the origin can never reach the spaceship. Explain why the straight line $c(t - \frac{c}{g}) = x$ defines a horizon.

Problem 2

The Lagrangian for a free electron-positron field is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \ , \tag{2}$$

where ψ is four-component column vector, $\bar{\psi} = \psi^\dagger \gamma^0$ is four-component row vector, and γ^μ are 4×4 matrices, called the γ-matrices. The γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \ , \tag{3}$$

for example $\gamma^0\gamma^0 = I$ and $\gamma^0\gamma^1 = -\gamma^1\gamma^0$. They also satisfy $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

a) We define a new matrix γ^5 as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{4}$$

Show that

$$(\gamma^5)^\dagger = \gamma^5 \ , \tag{5}$$
$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \ . \tag{6}$$

b) Consider a socalled *chiral* transformation

$$\psi \to e^{-i\alpha\gamma^5}\psi \ , \tag{7}$$
$$\psi^\dagger \to \psi^\dagger e^{i\alpha(\gamma^5)^\dagger} \ , \tag{8}$$

where $\alpha \in [0, 2\pi)$ is independent of the spacetime coordinates. How does $\bar{\psi}$ transform? For what values of m is the Lagrangian (2) invariant under a transformation (7)–(8)?

c) Calculate the conserved current that follows when (8) is a symmetry of the Lagrangian (2).

Problem 3

In this problem we consider gravitational waves in the weak-field limit. We set $c = 1$. The metric $g_{\mu\nu}$ is then written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; , \tag{9}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $h_{\mu\nu}$ is a small metric perturbation. This means that $|h_{\mu\nu}| \ll 1$. The first and higher partial derivatives of $h_{\mu\nu}$ are also small. Calculate all quantities below to *first order*. **Hint:** Calculating to first order implies that we can raise and lower indices using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. Thus $h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$. It then follows that $\partial_\alpha h^{\alpha\beta} = \partial^\alpha h_\alpha{}^\beta$.

a) Calculate the Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$.

b) Calculate the Riemann curvature tensor $R^\mu{}_{\beta\nu\gamma}$.

c) Show that the Ricci curvature tensor can be written as

$$R_{\mu\nu} = \frac{1}{2}\left[\partial_\nu\partial_\rho h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu + \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right] \; , \tag{10}$$

where $h = h^\rho{}_\rho$ and $\Box = -\partial_\rho\partial^\rho$.

d) Show that the Ricci scalar can be written as

$$R = \Box h + \partial_\mu\partial_\nu h^{\mu\nu} \; . \tag{11}$$

e) Consider a general infinitesimal coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu(x) \; . \tag{12}$$

Show that the transformed metric perturbation $h'_{\mu\nu}$ is given by

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \; . \tag{13}$$

It can be shown that $R^\mu{}_{\beta\nu\gamma}$ is invariant under the transformation (13). If one thinks of $h_{\mu\nu}$ as a tensor field defined on the flat Minkowski background, Eq. (13) can be considered a gauge transformation in analogy with electromagnetism.

f) We next define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \; . \tag{14}$$

It can be shown that Einstein's field equation in the vacuum expressed in terms of the derivatives of $\bar{h}_{\mu\nu}$ and $\bar{h} = \bar{h}^{\delta}{}_{\delta}$ is

$$\frac{1}{2}\left[\partial_\mu \partial_\delta \bar{h}^{\delta}{}_{\nu} + \partial_\nu \partial_\delta \bar{h}^{\delta}{}_{\nu} + \Box \bar{h}_{\mu\nu}\right] - \frac{1}{2}\eta_{\mu\nu}\partial^\delta \partial^\gamma \bar{h}_{\delta\gamma} \;\; = \;\; 0 \; . \tag{15}$$

We next consider a gauge transformation on the metric perturbation $\bar{h}^{\mu\nu}$, which is given by

$$\bar{h}'^{\mu\nu} \;\; = \;\; \bar{h}^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu}\partial_\alpha \xi^\alpha \; . \tag{16}$$

Explain why we can always impose the gauge condition

$$\partial_\mu \bar{h}'^{\mu\nu} \;\; = \;\; 0 \tag{17}$$

on the metric perturbation $\bar{h}'^{\mu\nu}$. This gauge is called the Lorentz gauge. Show that with this condition, the field equation reduces to

$$\Box \bar{h}'^{\mu\nu} \;\; = \;\; 0 \; . \tag{18}$$

g) Show that Eq. (18) has plane-wave solutions of the form

$$\bar{h}'_{\mu\nu} \;\; = \;\; A_{\mu\nu} e^{-ik_\alpha x^\alpha} \; , \tag{19}$$

where $A_{\mu\nu}$ are the constant components of a symmetric tensor and k_α are the components of a four wavevector k. Show that the wave vector is transverse in the Lorentz gauge. What is the condition on k?

h) Since $A_{\mu\nu}$ is symmetric there are 10 independent components. The gauge condition (17) is four equations leaving us with 6 independent components. The gauge condition (17) is not unique so the Lorentz gauge is really a class of gauges. We can use this residual freedom to impose further restrictions on $A_{\mu\nu}$. Choosing the socalled *transverse-traceless* (TT) gauge, one finds

$$A^{(TT)}_{\mu\nu}\eta^{\mu\nu} \;\; = \;\; 0 \; , \tag{20}$$

$$A^{(TT)}_{\mu\nu}\delta^{\nu}_{0} \;\; = \;\; 0 \; . \tag{21}$$

Let us consider a wave whose wavevector is of the form $k_\mu = (-\omega, 0, 0, \omega)$. Show that this choice of k in conjunction with transversality and Eq. (21) yields

$$A^{(TT)}_{z\nu} \;=\; 0\;. \tag{22}$$

Finally, use Eqs. (20)–(22) to show that we can write the matrix $A^{(TT)}_{\mu\nu}$ as

$$A^{(TT)}_{\mu\nu} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{(TT)}_{xx} & A^{(TT)}_{xy} & 0 \\ 0 & A^{(TT)}_{xy} & -A^{(TT)}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{23}$$

where $A^{(TT)}_{xx}$ and $A^{(TT)}_{xy}$ are two independent constant. They are of physical significance.

Congratulations, you have just found a gravitational wave propagating with the speed of light in the vacuum. Notice the similarity with electromagnetic waves. They were proposed by Henri Poincaré in 1905 and subsequently predicted in 1916 by Albert Einstein. Gravitational waves were first observed in 2015 by the LIGO collaboration as a result of a merger of two black holes of 29 and 36 solar masses about 1.3 billion light-years away. The 2017 Nobel prize in physics was awarded to Barry Barish, Kip Thorne, and Rainer Weiss for their fundamental work on gravitational waves both theoretically and observationally.

Useful formulas

$$j^\mu \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi\,, \tag{24}$$

$$\Gamma^\alpha{}_{\beta\gamma} \;=\; \frac{1}{2}g^{\alpha\mu}\left[\frac{\partial g_{\gamma\mu}}{\partial x^\beta} + \frac{\partial g_{\beta\mu}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu}\right]\,, \tag{25}$$

$$R^\alpha{}_{\mu\beta\nu} \;=\; \partial_\beta \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\beta\delta}\Gamma^\delta{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\delta{}_{\mu\beta}\,, \tag{26}$$

$$R_{\mu\nu} \;=\; \partial_\gamma \Gamma^\gamma_{\mu\nu} - \partial_\nu \Gamma^\gamma_{\mu\gamma} + \Gamma^\gamma_{\mu\nu}\Gamma^\delta_{\gamma\delta} - \Gamma^\gamma_{\mu\delta}\Gamma^\delta_{\nu\gamma}\,, \tag{27}$$

$$0 \;=\; R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\,, \tag{28}$$

$$g'_{\mu\nu} \;=\; \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}\,. \tag{29}$$