NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology

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Allowed tools: all

1. Gravitational waves.

a.) How many independent components has a metric perturbation $h_{\mu\nu}$ described by the wave equation in the harmonic gauge: (2 pt)

☐ 2
☐ 4
☐ 5
☐ 6
☐ 10
☐ 16

b.) Consider the gravitational wave produced by a binary system of two equal masses M on a circular orbit in the xy plane which is seen by three observers at large distance on the x axis, the y axis and the z axis. Determine the observed polarisiation by expressing the wave as $h_{\mu\nu} = \sum_a h^{(a)}\varepsilon^{(a)}_{\mu\nu}$, where $\varepsilon^{(a)}_{\mu\nu}$ is an appropriate basis of the polarisation states. (12 pts)

2. Schwarzschild black hole.

The Riemann tensor in Schwarzschild coordinates is

$$R_{0101} = \frac{2M}{r^3}, \qquad R_{0202} = R_{0303} = -\frac{M}{r^3}, \tag{1}$$

$$R_{1212} = R_{1313} = \frac{M}{r^3}, \tag{2}$$

$$R_{2323} = -\frac{2M}{r^3}, \tag{3}$$

all other elements which cannot be obtained by its (anti-) symmetry properties are zero.

a.) Show that the Riemann tensor in the inertial system of a freely falling observer has the same form as given above. (8 pts)

b.) The distance n^i of two freely falling particles changes as

$$\ddot{n}^i = R^i{}_{00j} n^j.$$

Consider now a cube with mass m assumed to be a rigid body of length L. What are the forces and the stresses acting on the cube at the distance r? [Hint: Consider in (a Newtonian picture) the force which has to counter-balance the gravitational acceleration

of a mass element dm of the rigid body.] (16 pts)
c.) Evaluate numerically the stress for values appropriate for a human. (6 pts)

3. Cosmology.
The static Einstein universe contains no radiation and has positive curvature.
a.) For a given value of Λ, there is a unique value of the matter density ρ_0 such that $\ddot{R} = 0$. Express ρ_0 and the scale factor R_0 through Λ. (6 pts)
b.) Consider a small perturbation, $\rho_m = \rho_0 + \delta\rho$. Use the Friedmann equation to show that the resulting change δR satisfies

$$\frac{\mathrm{d}^2\delta R}{\mathrm{d}t^2} = B\delta R \tag{4}$$

with B as constant. Is the Einstein universe stable or not? (10 pts)

4. Symmetries.
a.) Consider in Minkowski space a complex scalar field ϕ with Lagrange density

$$\mathscr{L} = s_1\frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi + s_2\frac{1}{4}\lambda(\phi^\dagger\phi)^2\,.$$

Name the symmetries of the Lagrangian and explain your choice for the signs s_1 and s_2. (6 pts)
b.) Consider in Minkowski space the following Lagrange density

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu,$$

where A_μ is the photon field, $F_{\mu\nu}$ the field-strength tensor, and j^μ an external current. Calculate the resulting change of the action under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$. Show that the action is invariant, if the current is conserved. (6 pts)
c.) Generalise now the two Lagrangians to a general space-time. Explain the general rules you apply. Is the procedure unique? (4 pts)