NTNU

Institutt for fysikk

Fakultet for fysikk, informatikk og matematikk

Solution to the exam in FY3452 GRAVITATION AND COSMOLOGY

Saturday May 19, 2012

Det finnes også en norsk versjon av dette eksamenssettet.

This solution consists of 8 pages.

Problem 1. Motion in an expanding universe

The Friedmann-Lemaître-Robertson-Walker metric can for $k = 0$ be expressed by the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{1}$$

when we use units where the speed of light $c = 1$. Geodetic motion can in general be derived from the Lagrange function

$$L = -\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu. \tag{2}$$

where the initial conditions must satisfy the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$ for massive particles, and $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ for massless particles (light). Here $\dot{}$ means differensiation with respect to egentime τ.

a) Which assumptions are behind the derivation of the line element (1), and the more general line element where $k \neq 0$?

That the universe is *homogeneous* (i.e. looks the same at every place) and *isotropic* (i.e. looks the same in every direction). For the line element (1) one in addition assumes that the spatial part of the universe (in the cosmic frame) is flat.

b) Find the Euler-Lagrange equations for motion in the Friedmann-Lemaître-Robertson-Walker geometry when $k = 0$.

In this case the Lagrange function becomes

$$L = -\frac{1}{2}\dot{t}^2 + \frac{1}{2}a(t)^2\left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2\right).$$

With the Euler-Lagrange equations

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} &= \frac{\partial L}{\partial t},\\ \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x},\\ &\text{etc},\end{aligned}$$

we find

$$\begin{aligned}-\frac{d}{d\tau}\dot{t} &= a(t)a'(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),\\ \frac{d}{d\tau}a(t)^2\,\dot{x} &= a(t)^2\left(\ddot{x} + \frac{2a'(t)}{a(t)}\dot{t}\dot{x}\right) = 0,\\ \frac{d}{d\tau}a(t)^2\,\dot{y} &= a(t)^2\left(\ddot{y} + \frac{2a'(t)}{a(t)}\dot{t}\dot{y}\right) = 0,\\ \frac{d}{d\tau}a(t)^2\,\dot{z} &= a(t)^2\left(\ddot{z} + \frac{2a'(t)}{a(t)}\dot{t}\dot{z}\right) = 0.\end{aligned} \tag{3}$$

c) The geodetic equation can in general be written in the form

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0. \tag{4}$$

Use the results of the previous point to find the connection coefficients $\Gamma^\mu{}_{\nu\lambda}$.

The answer is easily read out of (3)

$$\Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz} = a(t)a'(t), \tag{5}$$

$$\Gamma^x{}_{tx} = \Gamma^x{}_{xt} = \Gamma^y{}_{ty} = \Gamma^y{}_{yt} = \Gamma^z{}_{tz} = \Gamma^z{}_{zt} = \frac{a'(t)}{a(t)}. \tag{6}$$

d) The Lagrange function is in this case invariant under the transformation $x^i \to x^i + \epsilon$ for $x^i = x, y, z$. Use the Nöther theorem to find the corresponding conserved quantities. Show that the result is consistent with the geodetic equations you have found.

With $\delta x^j = \delta^j_i$ for $j = x,\ y,\ z$ we find

$$P^x = \frac{\partial L}{\partial \dot{x}} = a(t)^2\,\dot{x} \tag{7}$$

$$P^y = \frac{\partial L}{\partial \dot{y}} = a(t)^2\,\dot{y} \tag{8}$$

$$P^z = \frac{\partial L}{\partial \dot{z}} = a(t)^2\,\dot{z}. \tag{9}$$

The conservation laws $\frac{\partial}{\partial\tau}P_k = 0$ are *identical* to the Euler-Lagrange equations.

e) The Lagrange function is invariant under the transformation $\tau \to \tau + \epsilon$. What is the corresponding conserved quantity (which can be derived by use of the Nöther theorem)?

In this case the Lagrangian itself is the conserved quantity.

We insert $\delta t = \dot{t}$, $\delta x = \dot{x}$, $\delta y = \dot{y}$, $\delta z = \dot{z}$, and $\Lambda = L$ in the general formula, to find

$$H = \frac{\partial L}{\partial \dot{x}^\mu}\delta x^\mu - L = L. \tag{10}$$

f) Use the results above to find $\frac{dx^i}{dt}$ expressed by the starting values at $t = t_0$, and the function a.

We first find, since $P^i(t) = P^i(t_0)$,

$$\dot{x}^i(t) = \frac{a(t_0)^2}{a(t)^2}\dot{x}^i(t_0), \tag{11}$$

which however is not quite what was asked for, We next convert derivatives with repect to τ to derivatives with respect to t, using

$$\frac{dx^i}{dt} = \frac{d\tau}{dt}\frac{dx^i}{d\tau} = \frac{\dot{x}^i}{\dot{t}}.$$

Conservation of L is equivalent to constant normalization of the four-velocity u^ν,

$$u_\mu u^\mu = \dot{t}^2 - a^2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \begin{cases} 1, & \text{for massive particles} \\ 0, & \text{for massless particles.} \end{cases}$$

For massive particles we find that

$$\boldsymbol{v}^2 \equiv \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{\dot{t}^2} \equiv \frac{\dot{\boldsymbol{x}}^2}{1 + a^2\,\dot{\boldsymbol{x}}^2} \le \frac{1}{a^2}$$

Inverted

$$\dot{\boldsymbol{x}}^2 = \frac{\boldsymbol{v}^2}{1 - a^2\,\boldsymbol{v}^2}.$$

Hence we find

$$\dot{\boldsymbol{x}}_0 = \frac{\boldsymbol{v}_0}{\sqrt{1 - a_0^2\,\boldsymbol{v}_0^2}}, \quad \dot{\boldsymbol{x}} = \frac{a_0^2}{a^2}\,\dot{\boldsymbol{x}}_0, \quad \dot{\boldsymbol{x}}^2 = \frac{a_0^4}{a^4}\,\frac{\boldsymbol{v}_0^2}{1 - a_0^2\,\boldsymbol{v}_0^2},$$

and finally

$$\begin{aligned} \boldsymbol{v} &= \frac{\dot{\boldsymbol{x}}}{\sqrt{1 + a^2\,\dot{\boldsymbol{x}}^2}} = \frac{1}{\sqrt{1 + a^2\dot{\boldsymbol{x}}^2}}\,\frac{a_0^2}{a^2}\,\frac{\boldsymbol{v}_0}{\sqrt{1 - a_0^2\,\boldsymbol{v}_0^2}} \\ &= \frac{a_0^2}{\sqrt{a^4 + (a_0^2 - a^2)(a a_0 \boldsymbol{v}_0)^2}}\,\boldsymbol{v}_0. \end{aligned} \tag{12}$$

The massless result is obtained by taking the limiting case $(a_0\,\boldsymbol{v}_0)^2 = 1$, which gives

$$\boldsymbol{v} = \frac{a_0}{a}\,\boldsymbol{v}_0. \tag{13}$$

Problem 2. The Friedmann equations and consequences

The two Friedmann equations describing the dynamics of the universe can (in units where $c = 1$) be formulated as

$$G^0_0 = \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}G_N\,\varepsilon, \tag{14}$$

$$G^1_1 = \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2} + \frac{2a''}{a} = -8\pi G_N\,p, \tag{15}$$

where $'$ means differentiation with respect to cosmic time t, and G_N is Newton's constant. We have assumed an energy-momentum tensor of the form

$$T^\mu{}_\nu = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}. \tag{16}$$

a) Consistency of the Einstein equations $G^\mu{}_\nu = 8\pi G_N\,T^\mu{}_\nu$ in general impose a condition on the energy momentum tensor $T^\mu{}_\nu$. Which condition is it?

By construction the Einstein tensor is symmetric, $G^{\mu\nu} = G^{\nu\mu}$, and covariantly conserved, $G^{\mu\nu}{}_{;\mu} = 0$. Hence the energy-momentum tensor must also be symmetric, $T^{\mu\nu} = T^{\nu\mu}$, and (provided G_N is constant) covariantly conserved

$$T^{\mu\nu}{}_{;\mu} = 0. \tag{17}$$

b) In this problem you shall derive the condition by direct manipulation of equations (14) and (15): Differentiate equation (14) with respect to t, and next use equations (14) and (15) to eliminate all terms which involve $\frac{a''}{a}$ and $\frac{k}{a^2}$ from the result.

Differentiation of (14) gives

$$\frac{2a'a''}{a^2} - \frac{2a'^3}{a^3} - \frac{2ka'}{a^3} = \left(\frac{2a''}{a^2} - \frac{2a'^2}{a^2} - \frac{2k}{a^2}\right)\frac{a'}{a} = \frac{8\pi}{3}G_N\varepsilon'. \tag{18}$$

Now we use (15) to write

$$\frac{2a''}{a^2} = -\left(\frac{a'^2}{a^2} + \frac{k}{a^2}\right) - 8\pi G_N p,$$

and next (14) to write

$$-3\left(\frac{a'^2}{a^2}+\frac{k}{a^2}\right)=-8\pi G_N\varepsilon.$$

Inserted into equation (18) this expression becomes

$$-8\pi G_N\left(p+\varepsilon\right)\frac{a'}{a}=\frac{8\pi}{3}G_N\varepsilon',$$

equivalent to

$$\varepsilon'+3\frac{a'}{a}\left(p+\varepsilon\right)=0. \tag{19}$$

c) Show that the expression from the previous point can be written on the form

$$\frac{d}{dt}\varepsilon a^3+p\frac{d}{dt}a^3=0. \tag{20}$$

What is the physical interpretation of this equation?

We multiply equation (19) by a^3 to get equation (20). We may interprete εa^3 as the total energy U in a volume $V=a^3$. Equation (20) then says that

$$dU+pdV=0, \tag{21}$$

which compared with the thermodynamic identity

$$TdS=dU+pdV,$$

says that the universe (according to the Friedmann equations) expands in such a way that its entropy is unchanged (adiabatic expansion).

d) Assume the equation of state $p=w\varepsilon$, where w is a constant. Show that we then may use (20) to find a connection between $\varepsilon(t)/\varepsilon(t_0)$ and $a(t)/a(t_0)$. Give this connection.

We insert $p=w\varepsilon$ in equation (19) and divide by ε. The result is

$$\frac{\varepsilon'}{\varepsilon}+3(1+w)\frac{a'}{a}=\frac{d}{dt}\log\left(\varepsilon\, a^{3(1+w)}\right)=0, \tag{22}$$

which when integrated says that

$$\frac{\varepsilon(t)}{\varepsilon(t_0)}=\left(\frac{a(t_0)}{a(t)}\right)^{3(1+w)}. \tag{23}$$

e) Use the connection you found in the previous point to replace the right hand side of equation (14), to obtain an equation where $a(t)$ is the only time dependent quantity.

We insert equation (23) into (14) to get

$$\frac{a'(t)^2}{a(t)^2}+\frac{k}{a(t)^2}=C\,a(t)^{-3(1+w)},\quad \text{with } C=\frac{8\pi}{3}G_N\varepsilon(t_0)a(t_0)^{3(1+w)}. \tag{24}$$

f) Finally set $k=0$ and find the explicit solution of the equation you found in the previous point. Consider in particular the cases $w=\frac{1}{3}$, $w=0$, and $w=-1$.

For $w=\frac{1}{3}$ (hot matter) we get the equation $a'(t)^2=Ca(t)^{-2}$, or

$$a(t)\,a'(t)=\frac{1}{2}\frac{d}{dt}a(t)^2=\pm\sqrt{C},$$

with solution

$$a(t)=K\left(t-t_0\right)^{1/2}\quad \text{(big bang), or } a(t)=K\left(t_0-t\right)^{1/2}\quad \text{(big crunch)}. \tag{25}$$

For $w = 0$ (cold matter) we get the equation $a'(t)^2 = Ca(t)^{-1}$, or

$$a(t)^{1/2}\, a'(t) = \frac{2}{3}\frac{d}{dt} a(t)^{3/2} = \pm\sqrt{C},$$

with solution

$$a(t) = K\,(t - t_0)^{2/3} \quad \text{(big bang), or } a(t) = K\,(t_0 - t)^{2/3} \quad \text{(big crunch).} \tag{26}$$

For $w = -1$ (cosmological constant or vacuum energy) we get the equation $a'(t)^2 = C$, or

$$a'(t) = \pm\sqrt{C},$$

with solution

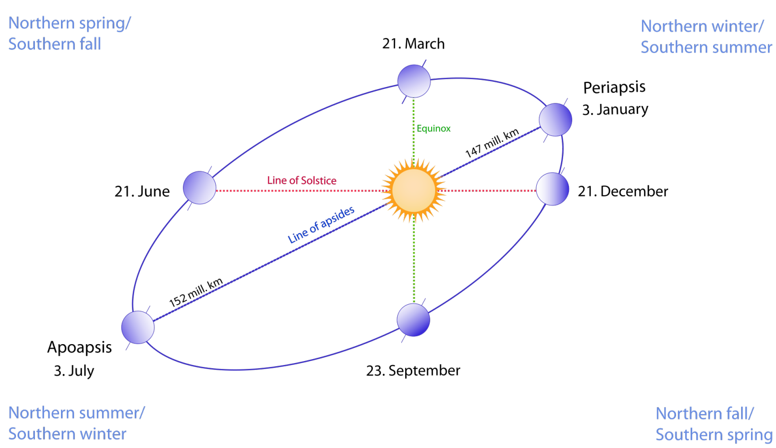
$$a(t) = K\,\mathrm{e}^{\sqrt{C}t} \quad \text{(cosmic inflation), or } a(t) = K\,\mathrm{e}^{-\sqrt{C}t} \quad \text{(cosmic deflation).} \tag{27}$$

Problem 3. Some astronomical facts

a) What is the distance from the earth to the Sun?

Roughly, 150 million kilometres (or 500 lightseconds).

More precisely, *aphelion* — the furthest *ap*art distance — is 152 098 232 kilometres, and *perihelion* is 147 098 290 kilometres. Perihelion occurs between january 2nd and 5th, aphelion between july 4th and 7th. This variation in distance means that the energy flux from the sun is about 7% higher i early january than early july. Good for us living on the northern hemisphere!



b) What is the speed of the earth around the Sun?

Roughly 30 kilometres per second.

More precisely, the average speed is 29.78 kilometres per second. To fulfill Kepler's 2nd law the speed is highest when the distance is smallest, making the winter halfyear shorter than the summer halfyear. Also nice!

c) What is the distance from Jupiter to the Sun?

Roughly 5 times the Sun–Earth distance.

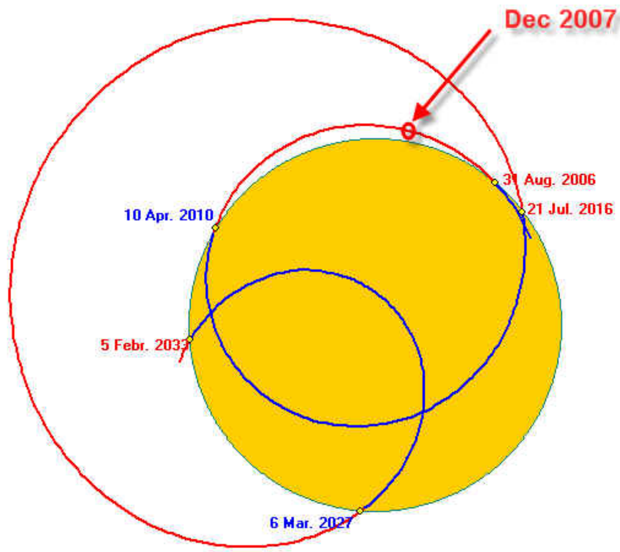
More precisely, it varies between 740 573 600 kilometres and 816 520 800 kilometres.

d) What is the radius of the Sun?

About 694 000 kilometres, i.e. 0.5% of the Sun-Earth distance.

e) Where is the center-of-mass of the solar system?

It depends a little on the positions of the other planets, but it is not very far from the center of the Sun, in particular for the moment.



Name	M/M_{J}	D/D_J	$(D\,M)/(D_J\,M_J)$
Mercury	0.0002	0.075	0.0000
Venus	0.0026	0.138	0.0004
Earth	0.0032	0.192	0.0006
Mars	0.0003	0.292	0.0001
Jupiter	1.0000	1.000	1.0000
Saturn	0.2996	1.835	0.5496
Uranus	0.0459	3.696	0.1698
Neptune	0.0541	7.781	0.3129

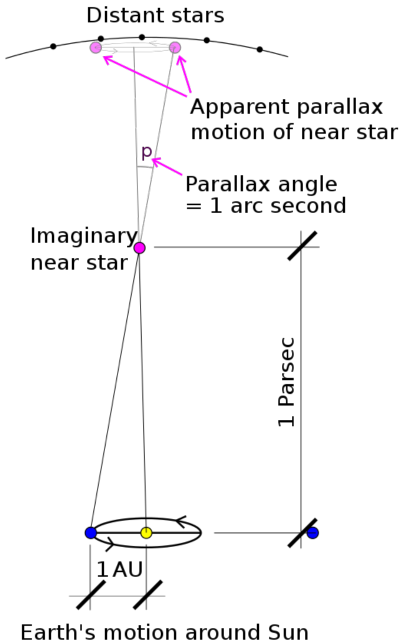
The masses and distances of the planets in the solar system, relative to the same quantities for Jupiter, are listed in the table above. The last column show the relative effect of each planet on the center-of-mass (with magnitude mass times distance, but with a varying direction). We note that the three outer planets, when they align, will have about the same effect as Jupiter alone. We conclude that the center-of-mass of the solar system varies within about one solar diameter from the senter of the sun.

f) What is the *astronomical unit* (a.u)?

The astronomical unit is $149\,597\,870\,700 \pm 3$ metres or $499.004\,783\,806\,1 \pm 0.000\,000\,01$ light-seconds, roughly the mean distance between the Earth and the Sun.

g) How long is a *lightyear*?

Exactly 9 460 730 472 580.8 kilometres, the distance travelled by light through vacuum in 31 557 600 seconds, or about 63 241.1 astronomical units.



h) How long is a *parsec*?

Roughly, a parsec is the distance from the Sun to an astronomical object which has a parallax angle of one arcsecond. I.e., one has

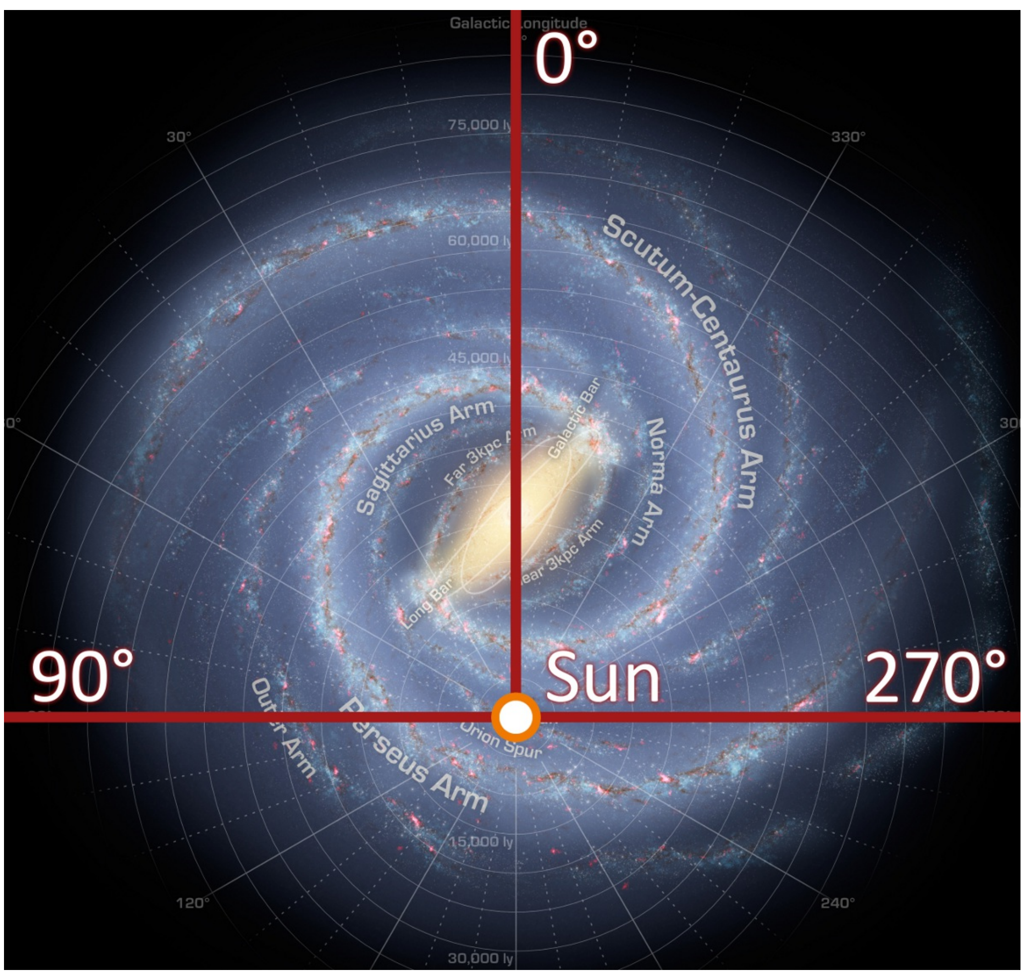
$$\begin{aligned} 1 \text{ parsec} &= \operatorname{arccot}\left(\frac{\pi}{360 \times 60 \times 60}\right) \text{ a.u.} = \frac{1\,296\,000}{2\pi} \text{ a.u.} \\ &= 206\,265 \text{ a.u.} \approx 3.26 \text{ lightyears.} \end{aligned}$$

i) What is the distance to the closest (next to the sun) star?

Our closest star (for the moment *Proxima Centauri*) is for the moment 4.2 lightyears away.

j) What is the distance from the sun to the center of our galaxy?

About 27 000 lightyears, cf. the figure below.



k) What is the size of our galaxy?

The visible part is about 100 000–120 000 lightyears in diameter.

l) What is the distance to the nearest spiral galaxy?

Our nearest spiral galaxy is the *Andromeda* galaxy, 2.6 million lightyears from the Earth.

m) What is the age of the universe?

The best current estimate is about 13.75×10^9 year.

Some expressions which *may* be of use

Euler-Lagrange equations

The Euler-Lagrange equations for a field theory described by the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a, x)$ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_a}. \tag{28}$$

The corresponding equations for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

Nöther's theorem

Assume the action is invariant under the continuous transformations $\varphi_a \to \varphi_a + \varepsilon\, \delta\varphi_a + \mathcal{O}(\varepsilon^2)$, more precisely that $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu + \mathcal{O}(\epsilon^2)$ under this transformation. Then there is an associated conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - \Lambda^\mu. \tag{29}$$

I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.