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Solution to the exam in
FY3452 GRAVITATION AND COSMOLOGY

Saturday august 11, 2012

This solution consists of 3 pages.

Problem 1. Faster-than-light (superluminal) motion?

Comparison of observed known spectral lines from the quasar 3C345 (with wavelength λ_{3C345}) with corresponding ones from earth (with wavelength λ_{earth}) show a considerable redshift

$$\frac{\lambda_{3C345}}{\lambda_{\text{earth}}} \equiv 1 + z \approx 1.595. \tag{1}$$

a) First assume that the observation (1) is entirely due to the expansion of the universe, and use it to estimate the distance between earth and 3C345 (i) when the radiation was emitted, and (ii) today. Both measured in cosmological comoving coordinates, i.e. a coordinate system where the cosmic background radiation is isotropic.

b) Next assume instead that the observation (1) is entirely due to a relative motion between earth and 3C345 in flat spactime (Minkowski space, with relative velocity v, and use it to determine v.

c) The largest anisotropy measured in the cosmic background radiation kan be explained by the earth's motion relative to the comoving coordinate system (where the cosmic background ratiation is isotropic). About which velocity, v_{earth}, do you think this motion has today?

d) In the real world one must expect the redshift (1) to be due to both the expansion of the universe, and the motions of 3C345 and earth in comoving coordinates. Estimate which uncertainty the latter motions have on the distances you found in point **a)**.

e) Observations of clouds in 3C345 have identified a cloud moving with angular velocity

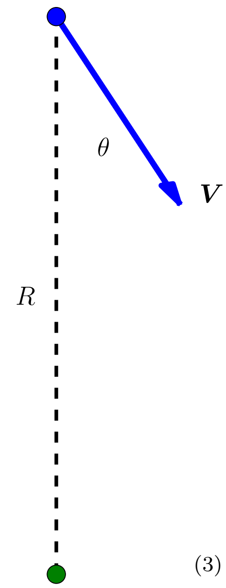
$$\omega = 2.3 \times 10^{-9} \, \frac{\text{radians}}{\text{year}} \tag{2}$$

relative to the center of 3C345. Assume that this cloud moves in a direction orthogonal to the line of sight. With which velocity $V_\perp$ must it move?

f) If you calculated correctly you should have found $V_\perp \gg c(!)$ in the previous point. This result has been used to argue that quasars cannot possibly be so far away as their redshift indicates. There is however another possible explanation, which you should investigate here.
Assume that the cloud move at an angle θ relative to the line of sight, as indicated in the figure to the right. The velocity $|\boldsymbol{V}|$ required to explain the observation (2) will depend on which angle θ we assume. Find this connection. What is the lowest value $|\boldsymbol{V}|$ may have?

Hint 1: To simplify the analysis you may assume that everything happens in Minkowski space.

Hint 2: Remember to take into account the time from the radiation is emitted until it is detected.



Oppgitt: The Hubble "constant" today (time t_0) is

$$H_0 \equiv \frac{d}{dt}\log(a(t))\Big|_{t=t_0} \approx \frac{0.721}{9.777\,752 \times 10^{10}\ \text{year}}. \tag{3}$$

Problem 2. Charged scalar field

The complex Klein-Gordon field can be used to model a collection of charged scalar particles. The dynamics is defined by the Langrangian

$$\mathcal{L} = \eta^{\mu\nu}\,(D_\mu\varphi)^*\,D_\nu\varphi - \kappa^2\varphi^*\,\varphi - \lambda\,(\varphi^*\,\varphi)^2 \tag{4}$$

where $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$ is the covariant derivative of electrodynamics, $|\kappa|^{-1}$ is a characteristic length parameter, and λ is an interaction parameter. We use the metric $\eta^{\mu\nu} = \text{diag}\,(1,-1,-1,-1)$.

a) What is the Euler-Lagrange equation for φ?

b) What is the Euler-Lagrange equation for φ^*?

c) Show that the Lagrangian (4) is invariant under global phase transformations,

$$\varphi(x) \to \varphi'(x) = \mathrm{e}^{\mathrm{i}\alpha}\,\varphi(x), \quad \varphi^*(x) \to \varphi^{*\prime}(x) = \mathrm{e}^{-\mathrm{i}\alpha}\,\varphi^*(x). \tag{5}$$

What are the infinitesimal versions of these transformations?

d) What is the conserved Nöther current resulting from this invariance?

e) What are the canonically conjugate momentum densities Π_φ and Π_{φ^*} of respectively the field φ and φ^*?

f) What is the Hamiltonian density $\mathcal{H}$ (energy density) of this model?

g) Assume that $A_0 \neq 0$ (but constant) and $\boldsymbol{A} = 0$. What is the lowest energy of this system? How does this result depend on the value of κ^2 (which you may assume to be both positive and negative)?

Problem 3. Indices, indices . . .

Many routine computations in General Relativity consists of index manipulations. The table below show some results of such manipulations. Which of these are (**R**) obviously right, (**W**) obviously wrong, and (**M**) maybe right—maybe wrong (depending on the quantities involved)?

a)	$g_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\gamma$	**W**	Free β and γ on right side only.
b)	$g_{\alpha\beta}\, a^\alpha b^\beta = g_{\alpha\beta}\, a^\alpha c^\beta$	**M**	If b and c have the same projection on a.
c)	$g_{\alpha\beta}\, a^\alpha b^\beta = g_{\beta\gamma}\, a^\gamma b^\beta$	**R**	Name of summation indices doesn't matter.
d)	$\Gamma^\alpha{}_{\alpha\gamma}\, a^\gamma = g_{\alpha\beta}\, a^\alpha b^\beta$	**M**	Strange equation, but not obviously wrong.
e)	$\Gamma^\alpha{}_{\beta\gamma}\, a^\alpha b^\beta c^\gamma = b^\alpha$	**W**	Two α's in upper position on left; free α on right.
f)	$\partial x^\alpha / \partial x^\beta = \delta^\alpha_\beta$	**R**	Basic property of partial derivatives.
g)	$\partial g_{\alpha\beta} / \partial x^\gamma = 0$	**M**	True if the metric is constant.
h)	$g_{\alpha\beta}\, \dfrac{\partial x^\alpha}{\partial x'^\gamma} \dfrac{\partial x^\beta}{\partial x'^\delta} = g_{\gamma\delta}\, \dfrac{\partial x^\gamma}{\partial x'^\alpha} \dfrac{\partial x^\delta}{\partial x'^\beta}$	**W**	Free α, β on left; free γ, δ on right.
i)	$g'_{\alpha\beta}\, a'^\alpha b'^\beta = g_{\alpha\beta}\, a^\alpha b^\beta$	**M**	True for some combinations of metrics and vectors.
j)	$a^\alpha \left(g_{\beta\gamma} b^\beta b^\gamma\right) = b^\gamma$	**W**	Free α on left; free γ on right.
k)	$\Gamma^\alpha{}_{\alpha\beta} = \Gamma^\beta{}_{\beta\beta}$	**W**	Trippel occurence of β on right.
l)	$g_{\alpha\beta} = \eta_{\alpha\beta}$	**M**	True in flat spacetime.
m)	$\xi^\mu \partial_\mu = \xi_\rho \partial^\rho$	**R**	Coordinated rise/lowering in index pairs is allowed.
n)	$g_{\mu\rho}\, \xi^\rho{}_{,\nu} = \xi_{\mu,\nu}$	**M**	True if the metric is constant.
o)	$g_{\mu\rho}\, \xi^\rho{}_{;\nu} = \xi_{\mu;\nu}$	**R**	True for metric compatible connections.

Some expressions which *may* be of use

Euler-Lagrange equations

The Euler-Lagrange equations for a field theory described by the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a, x)$ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_a}. \tag{6}$$

The corresponding equations for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

Nöther's theorem

Assume the action is invariant under the continuous transformations $\varphi_a \to \varphi_a + \varepsilon\, \delta\varphi_a + \mathcal{O}(\varepsilon^2)$, more precisely that $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu + \mathcal{O}(\epsilon^2)$ under this transformation. Then there is an associated conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - \Lambda^\mu. \tag{7}$$

I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.