NTNU – Trondheim
Norwegian University of
Science and Technology

Solutions Exam FY3452 Gravitation and Cosmology Spring 2016

Lecturer: Professor Jens O. Andersen
Department of Physics, NTNU
Telefon: 73593131

Tuesday May 31 2016
09.00-13.00

Aid:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler. In the problems, we use $c = G = 1$.

Problem 1

a) The formulas are

$$\begin{aligned} t' &= \underline{\underline{\gamma(t - vx)}} \,, & (1)\\ x' &= \underline{\underline{\gamma(x - vt)}} \,, & (2)\\ y' &= \underline{\underline{y}} \,, & (3)\\ z' &= \underline{\underline{z}} \,, & (4) \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$.

b) In the frame S', the four-momentum of the photon is $k'^{\mu} = \hbar(\omega', 0, k'_y, 0)$. These are now transformed to the frame S using the inverse transformations. These can be obtained by replacing v by $-v$. This yields

$$\begin{aligned}
\omega &= \gamma(\omega' + v k'_x) \\
&= \underline{\underline{\gamma\omega'}}\ . \qquad (5) \\
k_x &= \gamma(k'_x + v\omega') \\
&= \underline{\underline{\gamma v \omega'}}\ . \qquad (6) \\
k_y &= \underline{\underline{k'_y}}\ . \qquad (7) \\
k_z &= k'_z \\
&= \underline{\underline{0}}\ . \qquad (8)
\end{aligned}$$

c) The angle α is given by

$$\begin{aligned}
\tan\alpha &= \frac{k_y}{k_x} \\
&= \frac{k'_y}{\omega'}\frac{1}{\gamma v} \\
&= \frac{1}{\gamma v}\ , \qquad (9)
\end{aligned}$$

where we in the last line have used that $k'^2 = 0$ or $\omega' = k'_y$. An angle of $\frac{\pi}{4}$ yields the condition

$$\frac{1}{\gamma v} = 1\ . \qquad (10)$$

Solving this with respect to v, we find

$$v = \underline{\underline{\frac{1}{\sqrt{2}}}}\ . \qquad (11)$$

Problem 2

a) First consider $\Gamma^{\delta}_{\phi\phi}$. Since the only nonzero Christoffel symbol has $\delta = r$, this implies that $\alpha = r$ because the metric is diagonal. Thus one finds

$$\begin{aligned}
g_{rr}\Gamma^{r}_{\phi\phi} &= \frac{1}{2}\left[\frac{\partial g_{r\phi}}{\partial r} + \frac{\partial g_{r\phi}}{\partial r} - \frac{\partial g_{\phi\phi}}{\partial r}\right] \\
&= -\frac{1}{2} f'(r)\ . \qquad (12)
\end{aligned}$$

This implies

$$\Gamma^r_{\phi\phi} = \underline{\underline{-\frac{1}{2}f'(r)}} . \tag{13}$$

Next consider $\Gamma^\delta_{r\phi}$. Since the only nonzero Christoffel symbol has $\delta = \phi$, this implies that $\alpha = \phi$ since the metric is diagonal. This yields

$$\begin{aligned} g_{\phi\phi}\Gamma^\phi_{r\phi} &= \frac{1}{2}\left[\frac{\partial g_{\phi r}}{\partial \phi} + \frac{\partial g_{\phi\phi}}{\partial r} - \frac{\partial g_{r\phi}}{\partial \phi}\right] \\ &= \frac{1}{2}f'(r) . \end{aligned} \tag{14}$$

This implies

$$\Gamma^\phi_{r\phi} = \underline{\underline{\frac{1}{2}\frac{f'(r)}{f(r)}}} . \tag{15}$$

By symmetry $\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi}$.

b) The formula for the Ricci tensor is

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma_{\alpha\beta} - \partial_\beta \Gamma^\gamma_{\alpha\gamma} + \Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\gamma\delta} + -\Gamma^\delta_{\beta\gamma}\Gamma^\gamma_{\alpha\delta} , \tag{16}$$

This yields

$$\begin{aligned} R_{rr} &= \partial_r \Gamma^r_{rr} - \partial_r \Gamma^\gamma_{r\gamma} + \Gamma^\gamma_{rr}\Gamma^\delta_{\gamma\delta} - \Gamma^\delta_{r\gamma}\Gamma^\gamma_{r\delta} \\ &= -\partial_r \frac{1}{2}\frac{f'(r)}{f(r)} - \frac{1}{4}\frac{[f'(r)]^2}{f^2(r)} \\ &= \underline{\underline{-\frac{1}{2}\frac{f''(r)}{f(r)} + \frac{1}{4}\frac{[f'(r)]^2}{f^2(r)}}} . \end{aligned} \tag{17}$$

and

$$\begin{aligned} R_{\phi\phi} &= \partial_r \Gamma^r_{\phi\phi} + \Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi} - 2\Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r} \\ &= \underline{\underline{-\frac{1}{2}f''(r) + \frac{1}{4}\frac{[f'(r)]^2}{f(r)}}} . \end{aligned}$$

c) We need the inverse metric $g^{\alpha\beta}$ which is easily found by inversion of $g_{\alpha\beta}) = \mathrm{diag}(1, f(r))$. We find $g^{\alpha\beta} = \mathrm{diag}(1, 1/f(r))$. This yields

$$\begin{aligned} R &= g^{\alpha\beta}R_{\alpha\beta} \\ &= R_{rr} + \frac{1}{f(r)}R_{\phi\phi} \\ &= \underline{\underline{\frac{1}{2}\frac{[f'(r)]^2}{f(r)} - \frac{f''(r)}{f(r)}}} . \end{aligned} \tag{18}$$

d) Inserting $f(r) = r^n$, we find

$$R = \frac{1}{2} r^{2n-2} \left[2n - n^2 \right] . \tag{19}$$

We have $R = 0$ for either $n = \underline{\underline{0}}$ or $n = \underline{\underline{2}}$. The case $n = 2$ corresponds to flat Euclidean space, where the metric is expressed in polar coordinates. The case $n = 0$ corresponds to flat Euclidean space expressed in Cartesian coordinates. In the latter case, the coordinates are defined for the infinite strip $(r, \phi) \in [0, \infty] \times [0, 2\pi]$. One can trivially extend the coordinates to the entire plane.

Problem 3

a) The other coordinate singularities are given by the zeros of $1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}$. This yields the solutions

$$r_\pm = \underline{\underline{m \pm \sqrt{m^2 - \varepsilon^2}}} . \tag{20}$$

b) The null geodesics are given given by $ds^2 = 0$. Radial geodesics in addition has $d\theta = d\phi = 0$ and so we find

$$-(1 - f) d\bar{t}^2 + 2 f d\bar{t} dr + (1 + f) dr^2 = 0 . \tag{21}$$

One solution is $d\bar{t} = -dr$, which upon integration yields

$$\bar{t} + r = \text{constant} . \tag{22}$$

This is an *ingoing* light ray since f decreases as $\bar{t}$ increases.

c) By dividing Eq. (21) by dr and completing the square, one finds

$$\left[\frac{d\bar{t}}{dr} - \frac{f}{1-f} \right]^2 = \frac{1}{(1-f)^2} \tag{23}$$

or

$$\left[\frac{d\bar{t}}{dr} - \frac{f}{1-f} \right] = \pm \frac{1}{(1-f)} \tag{24}$$

Solving with respect to $\frac{d\bar{t}}{dr}$, we find $\frac{d\bar{t}}{dr} = -1$ (which corresponds to the solution above) and

$$\frac{d\bar{t}}{dr} = \underline{\underline{\frac{1+f}{1-f}}} . \tag{25}$$

Using the plot of $1-f$ and $1+f$ as functions of r, we conclude that $(1+f)/(1-f) > 0$ in region I and the null geodesic is therefore outgoing. In region II, on the other hand, $(1+f)/(1-f) < 0$ and so the null geodesic is incoming. In region III $(1+f)/(1-f) > 0$ so it is outgoing again. See Fig. 1.

d) This follows directly from properties of the null geodesics in region II and the fact that a particle is always inside the light cone. see Fig. 1. In fact, it can be shown that one can never reach the singularity in $r = 0$.

e) No, in region I, the one of the null geodesic is incoming and the other outgoing. Consequently the particles need not fall into the singularity at $r = 0$, see Fig. 1.

f) Inserting $\varepsilon^2 = \frac{3}{4}m^2$ into Eq. (20), we find

$$r_+ = \underline{\underline{\frac{3}{2}m}} \,. \tag{26}$$

$$r_- = \underline{\underline{\frac{1}{2}m}} \,. \tag{27}$$

The quantity is conserved

$$e = \left(1 - \frac{2M}{r} + \frac{\varepsilon^2}{r^2}\right)\frac{dt}{d\tau} \,. \tag{28}$$

Using that $\mathbf{u}\cdot\mathbf{u} = -1$

$$\left(1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right)^{-1} e^2 + \left(1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = -1 \,. \tag{29}$$

This can be rewritten as

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(-\frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right) \,. \tag{30}$$

Starting at rest at $r_+ = \frac{3}{2}m$ corresponds to $e = 0$. Thus the equation can be written as

$$\left(\frac{dr}{d\tau}\right) = \left(\frac{2m}{r} - 1 - \frac{\varepsilon^2}{r^2}\right)^{\frac{1}{2}} \tag{31}$$

This yields

$$\Delta\tau = \int_{\frac{1}{2}m}^{\frac{3}{2}m} \frac{dr}{\left(\frac{2m}{r} - 1 - \frac{\varepsilon^2}{r^2}\right)^{\frac{1}{2}}}$$

$$
\begin{aligned}
&= \int_{\frac{1}{2}m}^{\frac{3}{2}m} \frac{r dr}{\left(-(r-m)^2 + \frac{1}{4}m^2\right)^{\frac{1}{2}}} \\
&= m \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \frac{y+1}{\sqrt{-y^2 + \frac{1}{4}}} \, . \qquad (32)
\end{aligned}
$$

where we in the penultimate line have inserted the value $\varepsilon^2 = \frac{3}{2}m^2$ and where we in the last line have defined $y = (r-m)/m$. Finally, we change variable $y = \frac{1}{2}\cos x$ and we obtain

$$
\begin{aligned}
\Delta\tau &= m \int_0^{\pi} \left[1 + \frac{1}{2}\cos x\right] dx \\
&= \underline{\underline{\pi m}} \, . \qquad (33)
\end{aligned}
$$

This is the same result as for a Schwarzschild black hole where the particle starts at rest at the horizon $r = 2m$ and ends up at the singularity $r = 0$.

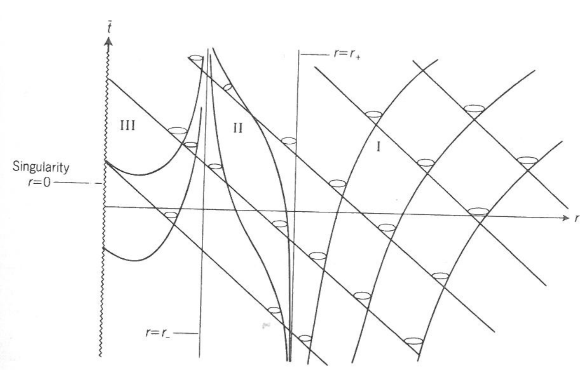

Figure 1: Null geodesics and light cones for a charged black hole.