NTNU – Trondheim
Norwegian University of
Science and Technology

Solutions Exam FY3452 Gravitation and Cosmology Fall 2016

Lecturer: Professor Jens O. Andersen
Department of Physics, NTNU
Phone: 73593131

Monday December 12 2016
09.00-13.00

Permitted examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler.

Problem 1

a) Taking the differentials, we obtain

$$\begin{aligned} dt' &= \gamma(dt - \frac{v}{c^2}dx) \\ &= \gamma dt\left(1 - \frac{vV_x}{c^2}\right) , \qquad (1) \\ dV_x' &= \frac{dV_x}{1-\frac{vV_x}{c^2}} + \frac{V_x - v}{(1-\frac{vV_x}{c^2})^2}\frac{v}{c^2}dV_x . \qquad (2) \end{aligned}$$

Dividing Eq. (2) by Eq. (1), we obtain

$$a_x' = \underline{\underline{\frac{1}{\gamma}\frac{a_x}{(1-\frac{vV_x}{c^2})^2} + \frac{1}{\gamma}\frac{V_x - v}{(1-\frac{vV_x}{c^2})^3}\frac{v}{c^2}a_x}} . \qquad (3)$$

If S' is the instantaneous rest frame, we have $v = V_x$ and Eq. (3) reduces to

$$a'_x = \underline{\underline{\gamma^3 a_x}} \,, \tag{4}$$

where we have used that $1 - \frac{vV_x}{c^2} = 1 - \frac{V_x^2}{c^2} = \frac{1}{\gamma^2}$.

b) Since $a'_x = g$, Eq. (4) can be written as

$$\frac{dV_x}{dt} = g\left(1 - \frac{V_x^2}{c^2}\right)^{\frac{3}{2}} \,. \tag{5}$$

or

$$\frac{dV_x}{\left(1 - \frac{V_x^2}{c^2}\right)^{\frac{3}{2}}} = gdt \,. \tag{6}$$

Changing variables, $V_x = c \sin u$, we obtain

$$\frac{cdu}{\cos^2 u} = gdt \,. \tag{7}$$

Integrating yields

$$c \tan u = gt + C \,, \tag{8}$$

where C is an integration constant.

$$\frac{V_x}{\sqrt{1 - \frac{V_x^2}{c^2}}} = gt + C \,. \tag{9}$$

Solving with respect to V_x, this finally yields

$$V_x(t) = \frac{gt + C}{\sqrt{\frac{1+(gt+C)^2}{c^2}}} \,. \tag{10}$$

$C = 0$ since $V_x(0) = 0$. Thus

$$V_x(t) = \underline{\underline{\frac{gt}{\sqrt{1 + \frac{g^2t^2}{c^2}}}}} \,. \tag{11}$$

The limiting velocity is $V_{\text{lim}} = \underline{\underline{c}}$ as seen from Eq. (11).

c) We have

$$\frac{d\tau}{dt} = \frac{1}{\gamma} = \frac{1}{\sqrt{1-\frac{V_x^2}{c^2}}} = \frac{1}{\sqrt{1+\frac{g^2t^2}{c^2}}} \tag{12}$$

Changing variables $t = \frac{c}{g}\sinh u$, we can write

$$d\tau = \frac{c}{g}du\ . \tag{13}$$

Integration yields

$$\tau = \frac{c}{g}\int_0^u du + C = \frac{c}{g}u + C = \frac{c}{g}\sinh^{-1}(\tfrac{g}{c}t) + K\ , \tag{14}$$

where K is an integration constant. $K = 0$ since $\tau(0) = 0$. This yields

$$t(\tau) = \underline{\underline{\frac{c}{g}\sinh(\tfrac{g}{c}\tau)}}\ . \tag{15}$$

d) Integrating Eq. (11), we find

$$x(t) = \frac{c^2}{g}\left[\sqrt{1+\frac{g^2t^2}{c^2}} - 1\right]\ , \tag{16}$$

where we have used that $x(\tau = 0) = x(t = 0) = 0$. Substituting Eq. (15) into Eq. (16), we finally obtain

$$x(\tau) = \underline{\underline{\frac{c^2}{g}\left[\cosh(\tfrac{g}{c}\tau) - 1\right]}}\ , \tag{17}$$

e) Taking the differentials of t and x yields

$$dt = \frac{1}{c}\sinh\left(\frac{gt'}{c}\right)dx' + \left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)\frac{g}{c}dt'\ , \tag{18}$$

$$dx = \cosh\left(\frac{gt'}{c}\right)dx' + c\left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)\frac{g}{c}dt'\ . \tag{19}$$

Inserting these expressions into the line element and using $dy = dy'$ and $dz = dz'$, we find

$$
\begin{aligned}
ds^2 &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \\
&= \underline{\underline{-c^2dt'^2\left(1+\frac{gx'}{c^2}\right)^2 + dx'^2 + dy'^2 + dz'^2}}\ ,
\end{aligned}
\tag{20}
$$

f) Since the line element is independent of time, the vector $\xi = (1, 0, 0, 0)$ is a Killing vector. The quantity $\boldsymbol{\xi}\cdot\boldsymbol{p}$ is a conserved quantity along a geodesic.

g) A stationary observer with spatial coordinates $(h, 0, 0)$ has four-velocity vector

$$
\begin{aligned}
\boldsymbol{u} &= \left(\left(1+\frac{gx'}{c^2}\right)^{-1}, 0, 0, 0\right) \\
&= \left(1+\frac{gx'}{c^2}\right)^{-1}\boldsymbol{\xi}\ .
\end{aligned}
\tag{21}
$$

The energy of a photon with four-momentum $\boldsymbol{p}$ and frequency ω is $\hbar\omega = -\boldsymbol{p}\cdot\boldsymbol{u}_{\rm obs}$. This yields

$$
\hbar\omega = -\left(1+\frac{gx'}{c^2}\right)^{-1}\boldsymbol{\xi}\cdot\boldsymbol{p}\ .
\tag{22}
$$

or

$$
\hbar\omega\left(1+\frac{gx'}{c^2}\right) = -\boldsymbol{\xi}\cdot\boldsymbol{p}\ .
\tag{23}
$$

The energy of a photon emitted at $x' = h$ is denoted by $\hbar\omega_h$ and the energy of the same photon absorbed at $x' = h$ is denoted by $\hbar\omega_0$. Eq. (23) then gives

$$
\omega_0 = \underline{\underline{\omega_h\left(1+\frac{gh}{c^2}\right)}}\ ,
\tag{24}
$$

since $\boldsymbol{\xi}\cdot\boldsymbol{p}$ is constant along the photon's geodesic.

According to the equivalence principle acceleration is equivalent to a gravitional field. The blueshift of the photon is an example of this principle.

Problem 2

a) Subtracting one-third of the first Friedman equation from the second Friedman equation gives

$$\ddot{a} = \underline{\underline{-\frac{4\pi}{3}a\rho_m + \frac{1}{3}a\Lambda}} \, . \tag{25}$$

where we have used that the pressure p vanishes.

b) For a time-independent solution, we have $\dot{a} = \ddot{a} = 0$. Equation (25), then yields

$$\rho_m^c = \underline{\underline{\frac{\Lambda}{4\pi}}} \, . \tag{26}$$

For a static solution the first Friedman equation reduces to

$$3\frac{1}{a_c^2} = 8\pi\rho_m^c + \Lambda \, , \tag{27}$$

or

$$a_c = \underline{\underline{\frac{1}{\sqrt{\Lambda}}}} \, . \tag{28}$$

c) We write $a = a_c + \delta a$. Note that $\dot{a} = \frac{d}{dt}\delta a$ and $\ddot{a} = \frac{d^2}{dt^2}\delta a$ since a_c is constant in time. For $p = 0$, the second Friedman equation can be rewritten as

$$2\ddot{a}a + \dot{a}^2 + 1 = \Lambda a^2 \, . \tag{29}$$

To first order in the perturbation, Eq. (29) reads

$$2a\frac{d^2}{dt^2}\delta a + 1 = \Lambda(a_c^2 + 2a_c\delta a) \, . \tag{30}$$

Using the result for a_c, we find

$$\frac{d^2}{dt^2}\delta a = \underline{\underline{\Lambda\delta a}} \, , \tag{31}$$

which corresponds to $B = \Lambda$. This is a second-order differential equation for δa, whose solution is

$$\delta a = A_1 e^{\sqrt{\Lambda}t} + A_2 e^{-\sqrt{\Lambda}t} \, , \tag{32}$$

where A_1 and A_2 are constants. The perturbation is growing and so the static Einstein universe is *unstable*. It is the sign of B that determines the stability of the solution. For $B < 0$, the solution for δa would involve trigonometric functions and so the universe would oscillate around the equilibrium solution.