NTNU – Trondheim
Norwegian University of
Science and Technology

Solutions Exam FY3452 Gravitation and Cosmology fall 2018

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09.00-13.00

Permitted examination support material:
Rottmann: Matematisk Formelsamling
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Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

Problem 1

a) Since the four-velocity vector $\mathbf{u} = (\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0)$ is normalized $\mathbf{u}\cdot\mathbf{u} = -c^2$, we find

$$\begin{aligned} \frac{dx}{d\tau} &= \sqrt{c^2\left(\frac{dt}{d\tau}\right)^2 - c^2} \\ &= \underline{\underline{c\sinh(\tfrac{g}{c}\tau)}}\,. \end{aligned} \tag{1}$$

Integration of $\frac{dt}{d\tau}$ gives

$$t(\tau) \;=\; \frac{c}{g}\sinh(\tfrac{g}{c}\tau) + t_0\ , \tag{2}$$

where t_0 is an integration constant. Using $t(0) = 0$, we find $t_0 = 0$ and

$$t(\tau) \;=\; \underline{\underline{\frac{c}{g}\sinh(\tfrac{g}{c}\tau)}}\ . \tag{3}$$

Integrating Eq. (1)

$$x(\tau) = \frac{c^2}{g}\cosh(\tfrac{g}{c}\tau) + x_0\ , \tag{4}$$

where x_0 is an integration constant. Using that $x(0) = 0$, we find $x_0 = -\frac{c^2}{g}$, which finally gives

$$x(\tau) \;=\; \underline{\underline{\frac{c^2}{g}\left[\cosh(\tfrac{g}{c}\tau) - 1\right]}}\ . \tag{5}$$

From Eqs. (3) and (5), we obtain

$$\left[x(\tau) + \frac{c^2}{g}\right]^2 - c^2t^2(\tau) = \frac{c^4}{g^2}\ , \tag{6}$$

which is the equation for a hyperbola.

b) The equation for the light ray is $x(t) = c(t - t_0)$. The position of the spaceship NTNU2018 is obtained from Eq. (6) and reads $x = \left(\sqrt{c^2t^2 + \frac{c^4}{g^2}} - \frac{c^2}{g}\right)$. Equating the two expressions, we find the time t when the signal is received. This yields

$$c(t - t_0) \;=\; \sqrt{c^2t^2 + \frac{c^4}{g^2}} - \frac{c^2}{g}\ . \tag{7}$$

Solving for t, we find

$$t \;=\; \underline{\underline{\frac{1}{2}\frac{t_0^2 - 2\frac{c}{g}t_0}{t_0 - \frac{c}{g}}}}\ . \tag{8}$$

This is a positive function in the interval $t_0 \in (0, \frac{c}{g})$. The time t diverges as $t_0 \to \frac{c}{g}$ from below showing that for $t_0 \geq \frac{c}{g}$ the light signal will never reach the spaceship. In Fig. 1, we have plotted the time t of the spaceship in units of $\frac{c}{g}$ (orange line) as a function of x in units of $\frac{c^2}{g}$. The red line is the worldline of a photon for $t_0 = \frac{1}{2}\frac{c}{g}$. The intercept of these curves gives the position and time of reception of a light signal. The yellow area shows the part of spacetime where no light signal can reach the spaceship. This area is bounded by the straight line $x = c(t - \frac{c}{g})$ and therefore acts as a horizon.

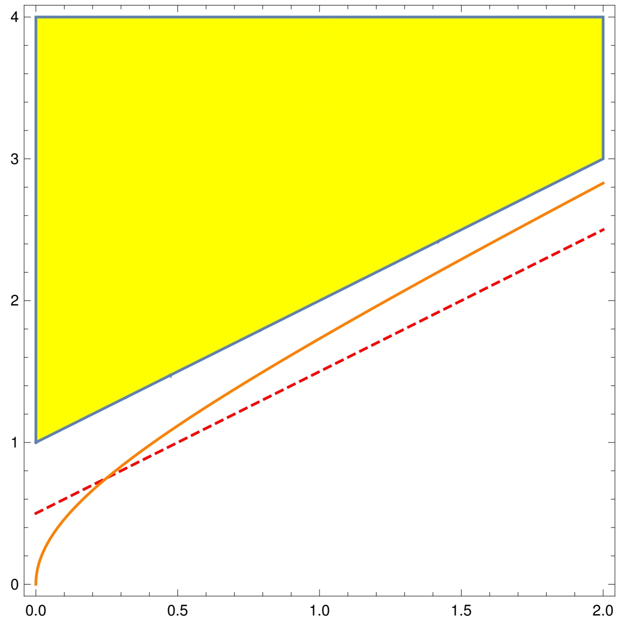

Figure 1: Hyperbolic motion and light signal.

Problem 2

a) The Hermitian cunjugate of γ^5 is

$$(\gamma^5)^\dagger \;=\; -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger\ .$$

Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ and that $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$, we can write

$$\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)(\gamma^0\gamma^0\gamma^0) = -i\gamma^0\gamma^3\gamma^2\gamma^1 \\ &= -i\gamma^0\gamma^1\gamma^3\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= \underline{\underline{\gamma^5}}\ .\end{aligned} \tag{9}$$

Thus γ^5 is Hermitean.

Since $\mu = 0, 1, 2$ or 3, γ^μ commute with one of the matrices in γ^5 and anticommute with the remaining three. We therefore get an overall minus sign as we pull γ^μ to the left and we find

$$\begin{aligned}\gamma^5\gamma^\mu &= (i\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^\mu \\ &= -\gamma^\mu(i\gamma^0\gamma^1\gamma^2\gamma^3) \\ &= -\gamma^\mu\gamma^5\ .\end{aligned} \tag{10}$$

In other words, γ^5 anticommutes with γ^μ:

$$\{\gamma^5, \gamma^\mu\} \;=\; \underline{\underline{0}}\ . \tag{11}$$

b) Since $\bar{\psi} = \psi^\dagger\gamma^0$, it transforms as

$$\bar{\psi} \;\; \rightarrow \;\; \psi^\dagger e^{i\alpha\gamma^5}\gamma^0 = \psi^\dagger\gamma^0 e^{-i\alpha\gamma^5} = \underline{\underline{\bar{\psi}e^{-i\alpha\gamma^5}}} \; , \tag{12}$$

where we have used that γ^5 anticommutes with γ^0. The kinetic term then transforms as

$$\begin{aligned} i\bar{\psi}\gamma^\mu\partial_\mu\psi &\rightarrow i\bar{\psi}e^{-i\alpha\gamma^5}\gamma^\mu\partial_\mu\psi e^{-i\alpha\gamma^5} \\ &= i\bar{\psi}e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\gamma^\mu\partial_\mu\psi \\ &= i\bar{\psi}\gamma^\mu\partial_\mu\psi \; , \end{aligned}$$

where we have used that γ^5 anticommutes with γ^μ. Hence the kinetic term is invariant under chiral transformations. The mass term transforms as

$$m\bar{\psi}\psi \;\; \rightarrow \;\; m\bar{\psi}e^{-2i\alpha\gamma^5}\psi \; , \tag{13}$$

which is not invariant. The Lagrangian is therefore invariant for $m = 0$.

c) Under infinitesimal chiral transformations we can write

$$\delta\psi \;\; = \;\; -i\alpha\gamma^5\psi \; , \tag{14}$$

$$\delta\bar{\psi} \;\; = \;\; -i\alpha\gamma^5\bar{\psi} \; , \tag{15}$$

which yields the deformations $\Delta\psi = \Delta\bar{\psi} = -i\gamma^5\psi$. Furthermore, the partial derivatives are

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \;\; = \;\; i\bar{\psi}\gamma^\mu \; , \tag{16}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} \;\; = \;\; 0 \; . \tag{17}$$

Using Eq. (24) in **Useful formulas**, the conserved current becomes

$$j^\mu \;\; = \;\; \underline{\underline{\bar{\psi}\gamma^\mu\gamma^5\psi}} \; . \tag{18}$$

This current is called the axial current since it is a pseuduvector under parity.

Problem 3

a) The Christoffel symbols are

$$\begin{aligned} \Gamma^\alpha{}_{\beta\gamma} &= \frac{1}{2}g^{\alpha\mu}\left[\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\right] \\ &= \frac{1}{2}\eta^{\alpha\mu}\left[\partial_\beta h_{\mu\gamma} + \partial_\gamma h_{\mu\beta} - \partial_\mu h_{\beta\gamma}\right] \\ &= \underline{\underline{\frac{1}{2}\left[\partial_\beta h^\alpha{}_\gamma + \partial_\gamma h^\alpha{}_\beta - \partial^\alpha h_{\beta\gamma}\right]}} \; , \end{aligned} \tag{19}$$

where we in the penultimate line have made the approximation $g^{\alpha\mu} = \eta^{\alpha\mu}$ since the derivative terms $\partial_\alpha g_{\beta\gamma} = \partial_\alpha h_{\beta\gamma}$ are of first order. This approximation is used in the remainder.

b) The Riemann curvature tensor is defined as

$$R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\beta\delta}\Gamma^\delta{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\delta{}_{\mu\beta} \,. \tag{20}$$

The products of the Christoffel symbols will be second order in $h_{\alpha\beta}$ and therefore we can write

$$\begin{aligned} R^\alpha{}_{\mu\beta\nu} &= \frac{1}{2}\partial_\beta \left[\partial_\nu h^\alpha{}_\mu + \partial_\mu h^\alpha{}_\nu - \partial^\alpha h_{\mu\nu}\right] - \frac{1}{2}\partial_\nu \left[\partial_\beta h^\alpha{}_\mu + \partial_\mu h^\alpha{}_\beta - \partial^\alpha h_{\mu\beta}\right] \\ &= \underline{\underline{\frac{1}{2}\left[\partial_\beta\partial_\mu h^\alpha{}_\nu + \partial_\nu\partial^\alpha h_{\beta\mu} - \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h^\alpha{}_\beta\right]}} \,. \end{aligned} \tag{21}$$

c) Contracting α and β, we find the Ricci curvature tensor

$$R_{\mu\nu} = \underline{\underline{\frac{1}{2}\left[\partial_\mu\partial_\rho h^\rho{}_\nu + \partial_\nu\partial_\rho h^\rho{}_\mu - \partial_\mu\partial_\nu h + \Box h_{\mu\nu}\right]}} \,, \tag{22}$$

where $h = h^\rho{}_\rho$ and $\Box = -\partial_\rho\partial^\rho$.

d) The Ricci scalar is

$$\begin{aligned} R &= \eta^{\mu\nu} R_{\mu\nu} \\ &= \underline{\underline{\Box h + \partial_\mu\partial_\nu h^{\mu\nu}}} \,. \end{aligned} \tag{23}$$

e) The coordinate transformation implies

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + \partial_\alpha \xi^\mu \,, \tag{24}$$

This can be inverted

$$\frac{\partial x^\mu}{\partial x'^\alpha} = \delta^\mu_\alpha - \partial_\alpha \xi^\mu \,, \tag{25}$$

which yields

$$\begin{aligned} g'_{\mu\nu} &= \eta_{\mu\nu} + h'_{\mu\nu} \\ &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= (\delta^\alpha_\mu - \partial^\alpha \xi_\mu)(\delta^\beta_\nu - \partial^\beta \xi_\nu)(\eta_{\alpha\beta} + h_{\alpha\beta}) \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \,, \end{aligned} \tag{26}$$

and therefore

$$h'_{\mu\nu} \;=\; \underline{\underline{h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu}} \,. \tag{27}$$

f) The transformed field is now

$$\bar{h}'^{\mu\nu} \;=\; \bar{h}^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu} \partial_\alpha \xi^\alpha \,. \tag{28}$$

Consider the equation

$$\partial_\mu \bar{h}'^{\mu\nu} \;=\; 0 \,. \tag{29}$$

From Eq. (28), this is equivalent to the equation

$$\partial_\mu \bar{h}^{\mu\nu} - \partial_\mu \partial^\mu \xi^\nu \;=\; 0 \,. \tag{30}$$

This equation always has a solution for any reasonably behaved $\bar{h}^{\mu\nu}$ and so we can always use the Lorentz gauge. The field equation is

$$\frac{1}{2}\left[\partial_\mu \partial_\delta \bar{h}^\delta{}_\nu + \partial_\nu \partial_\delta \bar{h}^\delta{}_\nu + \Box \bar{h}_{\mu\nu}\right] - \frac{1}{2}\eta_{\mu\nu}\partial^\delta \partial^\gamma \bar{h}_{\delta\gamma} \;=\; 0 \,. \tag{31}$$

Imposing the Lorentz gauge, trivially gives

$$\Box \bar{h}'^{\mu\nu} \;=\; 0 \,. \tag{32}$$

g) Inserting the plane wave into Eq. (32), we find

$$\Box A^{\mu\nu} e^{-ik_\alpha x^\alpha} \;=\; A^{\mu\nu} e^{-ik_\alpha x^\alpha} k^2 \,, \tag{33}$$

and is a solution for $k^2 = 0$, i.e. the wavevector is a null vector. The gauge condition yields

$$\partial_\mu A^{\mu\nu} e^{-ik_\alpha x^\alpha} \;=\; i k_\mu A^{\mu\nu} e^{-ik_\alpha x^\alpha} \,. \tag{34}$$

or $k_\mu A^{\mu\nu} = 0$, implying that the wave vector is transverse.

h) The gauge condition $A^{(TT)}_{\alpha\beta}\delta^\beta_0 = A^{(TT)}_{\alpha 0} = 0$ implies that the entries of first row and column of the matrix $A^{(TT)}$ vanish. Furthermore, transversality yields

$$\begin{aligned} k^\alpha A^{(TT)}_{\alpha\beta} &= k^0 A^{(TT)}_{0\beta} + k^z A^{(TT)}_{z\beta} = \omega(A^{(TT)}_{0\beta} + A^{(TT)}_{z\beta}) \\ &= \omega A^{(TT)}_{z\beta} = 0 \,, \end{aligned} \tag{35}$$

This implies that the entries of last row and column of the matrix A vanish. We are now left with four entries. Symmetry of A leaves us with $A_{xy}^{(TT)} = A_{yx}^{(TT)}$ and three independent entries. Finally, the traceless condition implies that $A_{xx}^{(TT)} + A_{yy}^{(TT)} = 0$. We can therefore write

$$A_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx}^{(TT)} & A_{xy}^{(TT)} & 0 \\ 0 & A_{xy}^{(TT)} & -A_{xx}^{(TT)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

where $A_{xx}^{(TT)}$ and $A_{xy}^{(TT)}$ are two independent constant.