NTNU – Trondheim
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# Solutions Exam FY3452 Gravitation and Cosmology fall 2019

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Thursday December 5 2019
15.00-19.00

Permitted examination support material:
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

## Problem 1

**a)** Since the argument $\frac{gt'}{c}$ of the hyperbolic functions is dimensionless, $g$ must have the dimension of acceleration, $\underline{\underline{m/s^2}}$. Taking the differentials of $t$ and $x$ yields

$$dt = \frac{1}{c}\sinh\left(\frac{gt'}{c}\right)dx' + \left(\frac{c}{g}+\frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)\frac{g}{c}dt' \,, \tag{1}$$

$$dx = \cosh\left(\frac{gt'}{c}\right)dx' + c\left(\frac{c}{g}+\frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)\frac{g}{c}dt' \,. \tag{2}$$

Inserting these expressions into the line element, we find

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 \\ &= \underline{\underline{-c^2dt'^2\left(1+\frac{gx'}{c^2}\right)^2 + dx'^2}} \ , \end{aligned} \tag{3}$$

**b)** The Christoffel symbols are defined as

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\mu}\left[\frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right] . \tag{4}$$

We therefore need the metric whose nonzero components are

$$g_{00} = -\left(1+\frac{gx'}{c^2}\right)^2 , \qquad g_{11} = 1 . \tag{5}$$

The inverse metric easily found by inversion of the metric $g_{\alpha\beta}$. This yields

$$g^{00} = -\left(1+\frac{gx'}{c^2}\right)^{-2} , \qquad g^{11} = 1 . \tag{6}$$

Since the metric is diagonal, the expression for $\Gamma^1_{00}$ collapses to

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2}g^{11}\left[-\frac{\partial g_{00}}{\partial x^1}\right] \\ &= = \underline{\underline{\left(1+\frac{gx'}{c^2}\right)\frac{g}{c^2}}} \ , \end{aligned} \tag{7}$$

where $x^1 = x'$ and $x^0 = ct'$.

The other nonzero Christoffel symbols are found in the same way,

$$\Gamma^0_{01} = \Gamma^0_{01} = \underline{\underline{\left(1+\frac{gx'}{c^2}\right)^{-1}\frac{g}{c^2}}} . \tag{8}$$

**c)** We first consider $R_{00}$, which is given by

$$\begin{aligned} R_{00} &= \partial_0\Gamma^0_{00} + \partial_1\Gamma^1_{00} - \partial_0\Gamma^0_{00} - \partial_0\Gamma^1_{01} + \Gamma^0_{00}\Gamma^{\delta}_{0\delta} + \Gamma^1_{00}\Gamma^{\delta}_{1\delta} - \Gamma^0_{0\delta}\Gamma^{\delta}_{00} - \Gamma^1_{0\delta}\Gamma^{\delta}_{01} \\ &= \partial\Gamma^1_{00} - \Gamma^1_{00}\Gamma^0_{01} = \frac{g^2}{c^4} - \frac{g^2}{c^4} = \underline{\underline{0}} . \end{aligned}$$

$R_{11}$ can be found in the same manner and vanishes as well, $R_{11} = 0$. This implies that

$$R = g^{\alpha\beta} R_{\alpha\beta} = \underline{\underline{0}} \, . \tag{9}$$

Since spacetime is flat, we know that the components of the Riemann curvature tensor all vanish identically, $R_{\alpha\beta\mu\nu} = 0$ in $S$. Since $R_{\alpha\beta\mu\nu}$ are the components of a tensor, they vanish in all coordinate systems. Since the Ricci curvature tensor and the Ricci scalar are obtained from the Riemann curvature tensor by contracting indices, they must vanish identically as well. The calculation in this problem shows this explicitly (for the diagonal components).

# Problem 2

**a)** Special relativity is based on

(a) The laws of physics take the same form in all inertial frames.

(b) The speed of light $c$ in vacuum is the same in all inertial frames.

**b)** The classification of the different types of curves is

(a) Timelike curves satisfy $|\frac{d\mathbf{x}}{cdt}| < 1$.

(b) Lightlike curves satisfy $|\frac{d\mathbf{x}}{cdt}| = 1$.

(c) Spacelike curves satisfy $|\frac{d\mathbf{x}}{cdt}| > 1$.

In Fig. 1, we have shown three straight lines with constant slope. The blue curve is a timelike curve, the black line is a lightlike curve, and the red line is a spacelike curve.

**c)** Taking differentials, we find

$$\Delta x' = \gamma(\Delta x - v\Delta t) \, , \tag{10}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \, . \tag{11}$$

If we now define $\Delta t = t_B - t_A > 0$ and $\Delta x = x_B - x_A > 0$ and consider $v_1 = c\frac{c\Delta t}{\Delta x}$. Since the events are timelike separated $\frac{c\Delta t}{\Delta x} < 1$. This implies $v_1 < c$ and it is therefore possible to choose a velocity $v$ such that $v < c$ and $v > v_1$. The first inequality

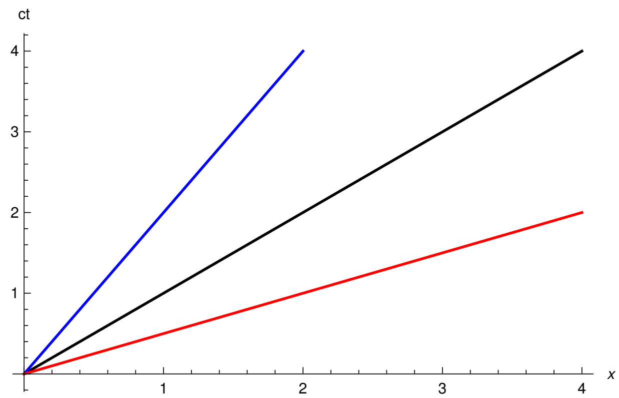


Figure 1: Different spacetime curves. See main text for classification.

guarantees that $v$ can be used to define an inertial frame $S'$, whle the second shows that

$$t'_B - t'_A \quad = \quad \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma\Delta t\left(1 - \frac{v}{c}\frac{\Delta x}{c\Delta t}\right) = \gamma\Delta t\left(1 - \frac{v}{v_1}\right) < 0 \tag{12}$$

In other words $t'_B > t'_A$. If $A$ is the cause of $B$, it must happen before $B$ in all inertial frames. The calculation shows that there exists a frame where $B$ is before $A$. Note that this happens precisely for spacelike separated events. Since signals cannot propagate faster than light, causality is not violated.

# Problem 3

**a)** The transformation of the field $\psi^\dagger$ follows from taking the Hermitian conjugate,

$$\psi^\dagger \quad \to \quad \psi^\dagger e^{-i\gamma^5\alpha}\,, \tag{13}$$

where we have used that $\gamma^5$ is Hermitian. This yields

$$\begin{aligned}\bar{\psi} &= \psi^\dagger\gamma^0 \\ &\to \psi^\dagger e^{-i\gamma^5\alpha}\gamma^0 \\ &= \psi^\dagger\gamma^0 e^{i\gamma^5\alpha} \\ &= \underline{\underline{\bar{\psi}e^{i\gamma^5\alpha}}}\,,\end{aligned} \tag{14}$$

where we have used that the anticommutation of $\gamma^0$ and$\gamma^5$.

**b)** Since $\gamma^5$ anitcommutes with $\gamma^\mu$, the kinetic term transforms as

$$\begin{aligned} \bar{\psi} i\gamma^\mu \partial_\mu \psi &\to \bar{\psi} e^{i\gamma^5\alpha} i\gamma^\mu \partial_\mu e^{i\gamma^5\alpha}\psi \; , \\ &= \bar{\psi} e^{i\gamma^5\alpha} e^{-i\gamma^5\alpha} i\gamma^\mu \partial_\mu \psi \\ &= \bar{\psi} i\gamma^\mu \partial_\mu \psi \; , \end{aligned} \tag{15}$$

i.e it is invariant. The mass term transforms as

$$m\bar{\psi}\psi \;\to\; m\bar{\psi} e^{2i\gamma^5\alpha}\psi \; . \tag{16}$$

This term is **not** invariant under chiral transformations. Hence the Dirac Lagrangian is invariant under chiral transformations only for $m = 0$.

# Problem 4

**a)** The cosmological principle is the notion that the spatial distribution of matter in the universe is homogeneous and isotropic about every point when viewed on large enough scales. Isotropy about a point meand that the universe looks the same in all directions and homogeneity means that it looks the same for all observers. Clearly this is an approximation, but a very useful starting point.

**b)** It follows from the cosmological principle that the spatial curvature of the universe is constant. From this result, it follows that the only possible spatial geometries are

(a) Space with zero curvature (flat, Euclidean)

(b) Space with constant positive curvature (closed, three-sphere)

(c) Space with constant negative curvature (open, hyperbolic)

The spatial metric can be summarized as

$$d\mathcal{L}^2 \;=\; \underline{\underline{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2}} \; , \tag{17}$$

where $k = 0$, $k = +1$, and $k = -1$, which correspond to the three cases listed above.