# NTNU Trondheim, Institutt for fysikk

**Examination for FY3452 Gravitation and Cosmology**
Contact: M. Kachelrieß, tel. 99890701
*Allowed tools: all*

**1. Gravitational waves.**
a.) How many independent components has a metric perturbation $h_{\mu\nu}$ described by the wave equation in the harmonic gauge: (2 pt)
☐ 2
☐ 4
☐ 5
☒ 6
☐ 10
☐ 16
b.) Consider the gravitational wave produced by a binary system of two equal masses $M$ on a circular orbit in the $xy$ plane which is seen by three observers at large distance on the $x$ axis, the $y$ axis and the $z$ axis. Determine the observed polarisiation by expressing the wave as $h_{\mu\nu} = \sum_a h^{(a)}\varepsilon^{(a)}_{\mu\nu}$, where $\varepsilon^{(a)}_{\mu\nu}$ is an appropriate basis of the polarisation states. (12 pts)

a.) The harmonic gauge imposes 4 constraints, leaving 6 independent components in the metric perturbation $h_{\mu\nu}$.

b.) In exercise sheet 9, we found

$$\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{8GM}{r} (\Omega R)^2 \begin{pmatrix} \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for the gravitational field of the binary. The general procedure to transform the amplitude into TT form is : Set the non-transverse components zo zero, and subtract half the resulting trace. In the TT gauge, we can use then $\bar{h}^{\rm TT}_{ij} = h^{\rm TT}_{ij}$.
For an observer along the $z$ direction, the results is already in the TT gauge, which we can express as

$$h^{\rm tt}_{\mu\nu} \propto \Re[(\varepsilon^{(1)}_{\mu\nu} - {\rm i}\varepsilon^{(2)}_{\mu\nu}) \exp(2{\rm i}\Omega t_r)].$$

This corresponds to a right-handed circularly polarized wave, $\varepsilon^{(-)}_{\mu\nu} = \varepsilon^{(1)}_{\mu\nu} - {\rm i}\varepsilon^{(2)}_{\mu\nu}$.
Next we consider an observer on the $x$ axis. Transforming to the TT form, we obtain

$$\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{8GM}{r} (\Omega R)^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\cos 2\Omega t_r & 0 \\ 0 & 0 & \cos 2\Omega t_r \end{pmatrix}.$$

This corresponds to a linearly polarized wave, $\propto \varepsilon^{(1)}_{\mu\nu}$ (with shifted diagonal elements).

**2. Schwarzschild black hole.**
The Riemann tensor in Schwarzschild coordinates is

$$R_{0101} = \frac{2M}{r^3}, \qquad R_{0202} = R_{0303} = -\frac{M}{r^3}, \tag{1}$$

$$R_{1212} = R_{1313} = \frac{M}{r^3}, \tag{2}$$

$$R_{2323} = -\frac{2M}{r^3}, \tag{3}$$

all other elements which cannot be obtained by its (anti-) symmetry properties are zero.
a.) Show that the Riemann tensor in the inertial system of a freely falling observer has the same form as given above. (8 pts)
b.) The distance $n^i$ of two freely falling particles changes as

$$\ddot{n}^i = R^i{}_{00j} n^j.$$

Consider now a cube with mass $m$ assumed to be a rigid body of length $L$. What are the forces and the stresses acting on the cube at the distance $r$? [Hint: Consider in (a Newtonian picture) the force which has to counter-balance the gravitational acceleration of a mass element $\mathrm{d}m$ of the rigid body.] (16 pts)
c.) Evaluate numerically the stress for values appropriate for a human. (6 pts)

The freely-falling frame and the standard Schwarschild coordinates are connnected by a Lorentz transformation. For a boost $\eta$, it is

$$R'_{0101} = \Lambda^\mu{}_0 \Lambda^\nu{}_1 \Lambda^\sigma{}_0 \Lambda^\rho{}_1 R_{\mu\nu\sigma\rho} = (\underbrace{\cosh^4\eta}_{0101} - \underbrace{2\cosh^2\eta\sinh^2\eta}_{1001,0110} + \underbrace{\sinh^4\eta}_{1010}) R_{0101} = R_{0101},$$

and similarly for the other non-zero elements.

b.) Inserting the Riemann tensor, it is

$$\ddot{n}^1 = \frac{2M}{r^3} n^1 \tag{4}$$

$$\ddot{n}^2 = -\frac{M}{r^3} n^2 \tag{5}$$

$$\ddot{n}^3 = -\frac{M}{r^3} n^3 \tag{6}$$

A volume element $\mathrm{d}m$ at the height $h$ above the center-of-mass (in direction $x^1$ would be acclerated by $a = 2M/r^3 h$ relative to the center-of-mass, if it could move freely. To prevent this, the force

$$\mathrm{d}F = a\mathrm{d}m = \frac{2}{M} r^3 h \mathrm{d}m$$

has to counter-act on the mass element. The total force along the plane is

$$F = \int_0^{L/2} \mathrm{d}LL^2\,\frac{2}{M}r^3\frac{m}{L^3} = \frac{mMl}{4r^3}$$

with the volume element $\mathrm{d}LL^2$ and the density $m/L^3$. The resulting stress $\sigma = -F/L^2$, and thus

$$\sigma_{\parallel} = -\frac{mM}{4Lr^3}, \qquad \sigma_{\perp} = \frac{mM}{8Lr^3},$$

c.) With $m = 80\,\mathrm{kg}$ and $L = 1\,\mathrm{m}$, the stresses are around

$$\sigma \sim 10^{15}\frac{\mathrm{dyn}}{\mathrm{cm}^2}\frac{M/M_\odot}{r/1\,\mathrm{km}}$$

(Compare with the normal pressure of Earth's atmosphere: $10^6\mathrm{dyn/cm}^2$.)

**3. Cosmology.**

The static Einstein universe contains no radiation and has positive curvature.
a.) For a given value of $\Lambda$, there is a unique value of the matter density $\rho_0$ such that $\ddot{R} = 0$. Express $\rho_0$ and the scale factor $R_0$ through $\Lambda$. (6 pts)
b.) Consider a small perturbation, $\rho_m = \rho_0 + \delta\rho$. Use the Friedmann equation to show that the resulting change $\delta R$ satisfies

$$\frac{\mathrm{d}^2\delta R}{\mathrm{d}t^2} = B\delta R \tag{7}$$

with $B$ as constant. Is the Einstein universe stable or not? (10 pts)

a.) We use first the acceleration equation,

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}\rho_m = 0$$

Thus $\rho_m = \Lambda/(4\pi G)$. The Friedmann equation gives

$$0 = H^2 = \frac{8\pi}{3}G\rho_m - \frac{1}{R^2} + \frac{\Lambda}{3} = (2+1)\frac{\Lambda}{3} - \frac{1}{R^2}$$

or $R = 1/\sqrt{\Lambda}$.

b.) The space-space part of the Einstein equation for a FLRW metric is given by

$$\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} = -\kappa P + \Lambda$$

Setting $P = 0$ and $k = +1$, it follows

$$2R\ddot{R} + \dot{R}^2 + 1 = \Lambda R^2.$$

From $R \propto 1/\sqrt{\rho_m}$, perturbations in matter lead to perturbations in the scale factor. Inserting $R = R_0 + \delta R$ and neglecting higher-order terms in $\delta R$, it follows

$$2R_0 \frac{\mathrm{d}^2 \delta R}{\mathrm{d}t^2} + 1 = \Lambda(R_0^2 + 2R_0 \delta R).$$

Using next $\Lambda R_0^2 = 1$, it follows

$$\frac{\mathrm{d}^2 \delta R}{\mathrm{d}t^2} = \Lambda \delta R.$$

Since $\Lambda > 0$, the solution is $\delta R = a \exp(\sqrt{\Lambda} t) + b \exp(\sqrt{-\Lambda} t)$, i.e. contains an exponentially growing term. Thus the Einstein universe is unstable.

**4. Symmetries.**

a.) Consider in Minkowski space a complex scalar field $\phi$ with Lagrange density

$$\mathscr{L} = s_1 \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi + s_2 \frac{1}{4} \lambda (\phi^\dagger \phi)^2 \,.$$

Name the symmetries of the Lagrangian and explain your choice for the signs $s_1$ and $s_2$. (6 pts)

b.) Consider in Minkowski space the following Lagrange density

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu,$$

where $A_\mu$ is the photon field, $F_{\mu\nu}$ the field-strength tensor, and $j^\mu$ an external current. Calculate the resulting change of the action under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$. Show that the action is invariant, if the current is conserved. (6 pts)

c.) Generalise now the two Lagrangians to a general space-time. Explain the general rules you apply. Is the procedure unique? (4 pts)

a. Space-time symmetries: Invariance under translation (4), Lorentz (3 boost and 3 rotations), scale and special conforma transformations (1+4, the latter 4 are probably unknow for you), i.e. in total 15 generators.

Internal symmetries: global U(1) (or SO(2)) invariance.

Choice of signs: The kinetic term $|\partial_t \phi|^2$ should be positive $\Rightarrow s_1 = +1$ (for a moostly negative metric); the energy should be bounded from below, thus $V = -s_2 \frac{1}{4} \lambda (\phi^\dagger \phi)^2$ requires $s_2 = -1$.

b.) The field-strength tensor is antisymmetric in $\partial_\mu A_\nu$ and thus gauge invariant. Hence the Lagrangian changes as

$$\delta \mathscr{L} = -j^\mu \partial_\mu \Lambda,$$

i.e. by a four-divergence. The change of the action follows as

$$\delta S = -\int \mathrm{d}^4 x j^\mu \partial_\mu \Lambda = \int \mathrm{d}^4 x \, [(\partial_\mu j^\mu) \Lambda - \partial_\mu (j^\mu \Lambda)] = -\int \mathrm{d}^4 x (\partial_\mu j^\mu) \Lambda.$$

Here, we used first the product rule and neglected then a boundary term. Thus $\delta S = 0$, if the photon couples to conserved current, $\partial_\mu j^\mu = 0$.

c.) Physical laws involving only quantities transforming as tensors on Minkowski space are valid on a curved spacetime performing the replacement

$$\{\partial_\mu, \eta_{\mu\nu}, \mathrm{d}^4x\} \to \{\nabla_\mu, g_{\mu\nu}, \mathrm{d}^4x\sqrt{|g|}\}\,.$$

For a scalar field, $\partial_\mu\phi = \nabla_\mu\phi$. Moreoverm the connection terms drop out in the field-strength tensor,

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha,$$

because it is antisymmetric. [More formally, we can identify completely antisymmetric tensors with differential-form for which differentiation without a connection is defined.] Thus the actions become simply

$$S = \int d^4x\sqrt{|g|}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^\dagger\partial_\mu\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2\right]$$

and

$$S = \int d^4x\sqrt{|g|}\left[-\frac{1}{4}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} - j^\mu A_\mu\right].$$

The procedure is not unique: First, we can add term which vanish in Minkowski space (e.g. $R^2\phi^2$ in case of a scalar field). Second, it may matter at which step we perform the replacement.