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~~Solution to the~~ Exam in
FY3464 QUANTUM FIELD THEORY I

Friday December 5th, 2008

This solution consists of 4 pages.

Problem 1. Processes in *QED*

Draw in all cases it is possible in Quantum Electrodynamics, (*QED*), the Feynman diagrams for all contributions of lowest non-trivial order for the processes below. In some cases there may exist Feynman diagrams, but the process is nevertheless impossible in vacuum. Indicate such cases, and explain briefly what makes the process impossible.

- **a)** $e^-\gamma \to e^-\gamma$
- **b)** $e^+\gamma \to e^+\gamma$
- **c)** $e^-e^+ \to \gamma$
- **d)** $e^-e^+ \to \gamma\gamma$
- **e)** $e^-e^+ \to e^-e^+$
- **f)** $e^-e^+ \to \mu^-\mu^+$
- **g)** $e^-\mu^+ \to e^+\mu^-$
- **h)** $\mu^- \to e^-\gamma$
- **i)** $\mu^- \to \mu^-e^-e^+$
- **j)** $\mu^-\gamma \to \mu^-e^-e^+$

Problem 2. Compton scattering in the center-of-mass system

In this problem you shall consider Compton scattering, $e^-\gamma \to e^-\gamma$, in more detail.

a) Draw the Feynman diagrams for all contributions of lowest non-trivial order to this process. Include all appropriate 4-momenta and indices. Assume that the incoming (resp. outgoing) electron has quantum numbers p, s (resp. p', s'), and that the incoming (resp. outgoing) photon has quantum numbers k, r (resp. k', r'). Further introduce $q = p + k$ og $q' = p - k'$.

b) Use the Feynman rules in the Appendix to write down the corresponding algebraic contributions to the scattering amplitude $\mathcal{M}_{fi}$.

c) Use "natural units" (where $\hbar = c = \varepsilon_0 = \mu_0 = 1$) and assume that the process is considered from the center-of-mass system where the incoming photon has frequency ω, and is scattered in the (x, z)-plane. I.e., so that $k = (\omega, 0, 0, \omega)$ and $k' = (\omega, \omega \sin\vartheta, 0, \omega \cos\vartheta)$.

Write down the corresponding 4-momenta p and p' in component form.

d) Find two real polarization 4-vectors e_r for the incoming photon, and two real polarization 4-vectors e'_r for the outgoing photon such that

$$k_\mu e^\mu_\mathrm{r} = p_\mu e^\mu_\mathrm{r} = 0, \quad k'_\mu e'^\mu_\mathrm{r} = p_\mu e'^\mu_\mathrm{r} = 0, \tag{1}$$

and such that they are orthonormalized,

$$\eta_{\mu\nu} e^\mu_\mathrm{r} e^\nu_\mathrm{s} = -\delta_\mathrm{rs}, \quad \eta_{\mu\nu} e'^\mu_\mathrm{r} e'^\nu_\mathrm{s} = -\delta_\mathrm{rs}. \tag{2}$$

e) Use the properties for e_r and e'_r above, plus the fact that $u(p, s)$ is a solution of the Dirac equation, $(\not{p} - m)\, u(p, s) = 0$, to simplify the scattering amplitudes as much as possible.

f) Use dimensional analysis and qualitative information from the Feynman diagrams to make an order-of-magnitude estimate of the total scattering cross section, in the special case where the incoming photon has very low frequency, $\omega \ll m$. I.e. determine which algebraic combination of physical parameters the cross section must depend on, and evaluate the value of this combination in SI-units.

g) The cross-section for unpolarized electrons is obtained by averaging over the spin states (s) of the incoming electron, and summing over the spin states (s') of the outgoing electron. The squared amplitude $\sum_{ss'} |\mathcal{M}_{fi}|^2$ may then be expressed as a sum of traces over γ-matrices (with prefactors).

Write down the expressions for all the traces involved in this sum. You don't have to evaluate the traces.

h) Evaluate the trace

$$T^{\mathrm{aa}} \equiv \mathrm{Tr}\left\{\not{e}'\, \not{k}\, \not{e}\, \not{p}\, \not{e}\, \not{k}\, \not{e}'\, \not{p}'\right\}, \tag{3}$$

where the 4-vectors involved have the properties of eq. (1-2), and $k_\mu k^\mu = k'_\mu k'^\mu = 0$.

Some physical constants:

Planck constant	$\hbar = 1.054\,572\,66(63) \times 10^{-34}$ J s
Speed of light	$c = 299\,792\,458$ m s^{-1}
Electron mass	$m = 9.1096 \times 10^{-31}$ kg
Fine structure constant	$\alpha = e^2/4\pi\varepsilon_0\hbar c = 1/1/137.035\,989\,5(61)$

Problem 3. Electrodynamics with a dynamical polarization field

Consider the model defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{4}\mathcal{P}_{\mu\nu}\mathcal{P}^{\mu\nu} + \frac{1}{2}M^2\mathcal{P}_\mu\mathcal{P}^\mu - \frac{1}{2}\varepsilon_r\,\mathcal{F}_{\mu\nu}\mathcal{P}^{\mu\nu}, \tag{4}$$

where $\mathcal{F}_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ and $\mathcal{P}_{\mu\nu} = \partial_\nu \mathcal{P}_\mu - \partial_\mu \mathcal{P}_\nu$. We use natural units and assume four space-time dimensions.

a) The action $S = \int \mathrm{d}^4x\,\mathcal{L}$ must be dimensionless. In that case, what is the mass dimension (inverse of the length dimension) of the fields A_μ and $\mathcal{P}_\mu$, and of the parameter ε_r?

The Lagrangian must be of dimension *mass*^{4}, hence $\mathcal{F}^{\mu\nu}$ and $\mathcal{P}^{\mu\nu}$ must be of dimension *mass*^{2}. This implies that A_μ and $\mathcal{P}_\mu$ must be of dimension *mass* (consistent with M being a mass parameter), and ε_r must be dimensionless.

b) Show that

$$\begin{aligned}\frac{\partial\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}}{\partial\,(\partial_\alpha A_\beta)} &= \frac{\partial\,\{\eta^{\mu\rho}\eta^{\nu\sigma}\,(\partial_\nu A_\mu - \partial_\mu A_\nu)\,(\partial_\sigma A_\rho - \partial_\rho A_\sigma)\}}{\partial\,(\partial_\alpha A_\beta)} \\ &= 4\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = -4\mathcal{F}^{\alpha\beta}.\end{aligned} \tag{5}$$

We get a nonzero result when $(\alpha,\beta) = (\nu,\mu)$, $(\alpha,\beta) = (\mu,\nu)$, $(\alpha,\beta) = (\sigma,\rho)$, or $(\alpha,\beta) = (\rho,\sigma)$. In the first case we get a contribution

$$\eta^\alpha{}_\nu\,\eta^\beta{}_\mu\,\eta^{\mu\rho}\eta^{\nu\sigma}\,(\partial_\sigma A_\rho - \partial_\rho A_\sigma) = \eta^{\beta\rho}\,\eta^{\alpha\sigma}\,(\partial_\sigma A_\rho - \partial_\rho A_\sigma) = \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = -\mathcal{F}^{\alpha\beta}.$$

By the same calculation we find that the three other cases give the same contribution, alltogether $-4\,\mathcal{F}^{\alpha\beta}$.

c) Find the equations of motion (Euler-Lagrange equations) for A_β and $\mathcal{P}_\beta$.

The calculations of the previous point show that

$$\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left(-\frac{1}{4}\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\right) = \mathcal{F}^{\alpha\beta}.$$

Likewise we find

$$\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left(-\frac{1}{2}\varepsilon_r\mathcal{F}_{\mu\nu}\,\mathcal{P}^{\mu\nu}\right) = \varepsilon_r\mathcal{P}^{\alpha\beta},$$

and

$$\frac{\partial}{\partial(\partial_\alpha \mathcal{P}_\beta)}\left(-\frac{1}{4}\mathcal{P}_{\mu\nu}\,\mathcal{P}^{\mu\nu}\right) = \mathcal{P}^{\alpha\beta},$$

and

$$\frac{\partial}{\partial(\partial_\alpha \mathcal{P}_\beta)}\left(-\frac{1}{2}\varepsilon_r\mathcal{F}^{\mu\nu}\,\mathcal{P}_{\mu\nu}\right) = \varepsilon_r\mathcal{F}^{\alpha\beta}.$$

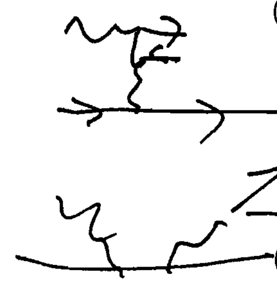
Hence the Euler-Lagrange equation

$$\partial_\alpha\frac{\partial\,\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial\,\mathcal{L}}{\partial A_\beta}$$

$\gamma \mu \to \mu e^+ e^-$

becomes

$$\partial_\alpha \left(\mathcal{F}^{\alpha\beta} + \varepsilon_\text{r} \mathcal{P}^{\alpha\beta} \right) = 0. \tag{6}$$



And the Euler-Lagrange equation

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \mathcal{P}_\beta)} = \frac{\partial \mathcal{L}}{\partial \mathcal{P}_\beta}$$

becomes

$$\partial_\alpha \left(\mathcal{P}^{\alpha\beta} + \varepsilon_\text{r} \mathcal{F}^{\alpha\beta} \right) = M^2 \mathcal{P}^\beta. \tag{7}$$

d) Show that $\partial_\beta \mathcal{P}^\beta = 0$ when $\mathcal{P}^\beta$ solves the Euler-Lagrange equations.

We operate with ∂_β on equation (7) and use the fact that $\partial_\beta \partial_\alpha$ is symmetric under the interchange $\alpha \rightleftarrows \beta$ while $\mathcal{F}^{\alpha\beta}$ and $\mathcal{P}^{\alpha\beta}$ are antisymmetric. Hence $\partial_\beta \partial_\alpha \mathcal{F}^{\alpha\beta} = 0$ and $\partial_\beta \partial_\alpha \mathcal{P}^{\alpha\beta} = 0$. With ε_r and M^2 constant this leads to $M^2\, \partial_\beta \mathcal{P}^\beta = 0$. With $M^2 \neq 0$ this implies $\partial_\beta \mathcal{P}^\beta = 0$.

e) Find the canonical conjugate momentum density $\Pi^\mu_\mathcal{P}$ of the field $\mathcal{P}_\mu$.

From the previous results we find

$$\Pi^\mu_\mathcal{P} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \mathcal{P}_\mu\right)} = \left(\mathcal{P}^{0\mu} + \varepsilon_\text{r} \mathcal{F}^{0\mu} \right). \tag{8}$$

f) Show that the Lagrangian (4) is invariant under the transformations

$$A_\mu(x) \rightarrow A_\mu(x) + \epsilon\, \partial_\mu \Lambda(x) \tag{9}$$

for any differentiable function $\Lambda(x)$. Here ϵ is a constant.

Under the transformation above we find that

$$\mathcal{F}_{\mu\nu} \longrightarrow \mathcal{F}_{\mu\nu} + \left(\partial_\nu \partial_\mu - \partial_\mu \partial_\nu \right) \Lambda = \mathcal{F}_{\mu\nu}, \tag{10}$$

since the order of differentiation does not matter (unless Λ is an *extremely pathological* function). $\mathcal{L}$ is invariant under the transformation (9) since $\mathcal{F}_{\mu\nu}$ is.

g) Use the Nöther procedure to find the corresponding conserved current J^μ_Λ.

We find the Nöther current to be

$$J^\alpha_\Lambda = \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha A_\beta\right)}\, \delta A_\beta = \left(\mathcal{F}^{\alpha\beta} + \varepsilon_\text{r} \mathcal{P}^{\alpha\beta} \right) \partial_\beta \Lambda. \tag{11}$$